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When winning is the only thing: pure strategy Nash equilibria in a three-candidate spatial voting model

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Abstract It is well known that there are no pure strategy Nash equilibria (PSNE) in the standard three-candidate spatial voting model when candidates maximize their share of the vote. When all that matters to the candidates is winning the election, however, we show that PSNE do exist. We provide a complete characterization of such equilibria and then extend our results to elections with an arbitrary number of candidates.

1 Introduction

Among the many extensions of Hotelling's (1929) classic introduction to the spatial location of firms is contained the well-known result that pure strategy Nash equilibria (PSNE) exist only when the number of firms differs from three (Eaton and Lipsey 1975). An alternative direction of research, first suggested by Hotelling himself, is the application of the spatial model to the location of candidates for political office.[1] Cox (1987) provides a thorough treatment of voting equilibria

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[1] Cox (1985, 1990), Greenberg and Shepsle (1987), Myerson and Weber (1993), Myerson (1999), Osborne (1993, 1995), Palfrey (1984), and Shepsle (1991) all analyze multicandidate elections under various voting procedures.

under various voting schemes with unidimensional platforms, nonstrategic voters, and vote-maximizing candidates. He demonstrates that PSNE fail to exist whenever there are at least three, and an odd number of, candidates. It is now recognized, however, that candidates may have other objectives than vote maximization, which is the natural counterpart in the political realm to the assumption that firms maximize their market share. Combining these two directions of research, we show that if winning is the candidates' only objective, then PSNE do exist in three-candidate or three-party elections. Although this result is straightforward, it does not appear to have been previously recognized in the literature.

We characterize the entire set of PSNE when there are three candidates, each maximizing her expected probability of winning the election. We show that in all of the PSNE, the winner is the sole candidate on her half of the political spectrum and, therefore, the winner does not locate at the position most preferred by the median voter.[2] Furthermore, the additional candidate on the losing half of the political spectrum may serve as an election spoiler for the other candidate on her side of the political spectrum (a result that may reflect some recent election experiences).

When there are more than three candidates, there are many possible PSNE; however, when the winner is the most extreme candidate on one side of the spectrum, the conditions for PSNE are closely related to those for the three-candidate case. In these equilibria, it is possible for the number of candidates to approach infinity without altering the general structure of the equilibrium set. There are other equilibria, however, and a PSNE whereby the winner adopts the platform most preferred by the median voter is obtainable. We also show how an increase in the number of candidates allows the winner to adopt a more extreme position. In particular, as the number of candidates increases from two to three, the winner must adopt a more extreme position, and as the number of candidates increases further, the distance between platforms can expand to encompass the entire political spectrum and the winner's position can approach the most extreme reaches of the spectrum.

2 Pure strategy Nash equilibria in three-candidate elections

The voting environment we have in mind is easily summarized. There exists a continuum of voters whose measure is normalized to one. The symmetric, single-peaked preferences of these voters are defined along a single dimension, and it is common knowledge that they are distributed uniformly along the unit interval.[3] Voters have no strategic incentives and, therefore, each casts her vote for the candidate whose platform is closest to her most preferred platform. If two or more

[2] Building on the well-known result that plurality voting leads to a distribution of platforms and does not lead to a median voter result (Eaton and Lipsey 1975; Denzau et al. 1985), Cox (1987) considers various voting schemes and shows that plurality voting is the only one that is not "centrist". Under all of these schemes, however, candidates are assumed to maximize their share of the vote.

[3] If the candidates were uncertain as to the prior distribution of voters, then the winner could conceivably deviate from certain configurations to increase their possible vote share. We leave as a topic for further research the role of differing types of uncertainty, and higher order beliefs, in reducing the multiplicity of PSNE and perhaps, in certain cases, precluding their existence.

candidates declare the same platform, then the candidates split equally all votes garnered by their platform.

Each candidate's objective is to maximize her expected chance of winning the election. Hence, candidates prefer winning the election to losing the election, but they do not otherwise care about their share of the vote. The winner of the election is the candidate who receives a plurality of the vote. In the case of a tie, the winner is selected randomly among all the candidates receiving the plurality share.

To see that a PSNE can exist when each of the three candidate's sole objective is to win the election, consider the platforms of $\{0.3, 0.4, 0.7\}$. The shares of the vote are then $\{0.35, 0.20, 0.45\}$. When candidates maximize their share of the vote, this configuration of platforms is not a PSNE, in part, because the rightmost candidate can increase her share by choosing any platform between 0.4 and 0.7. When candidates only care about winning the election, however, this configuration is a PSNE. Neither the leftmost nor the middle candidate can unilaterally deviate and capture a plurality of the vote, and the rightmost candidate has no incentive to deviate as she currently wins the election.

With Proposition 1 below, we provide a complete characterization of the PSNE in a three-candidate election in which winning the election is all that matters to the candidates. In our characterization, the winner's platform is greater than one-half. A symmetric set exists when the winner's location is less than one-half. Let the three candidates be labeled A, B, and C. Candidates simultaneously choose a platform, denoted by $P(x)$ where $P(x) \in [0, 1]$ for all $x \in \{A, B, C\}$. Without loss of generality, order the candidates so that $P(A) \leq P(B) \leq P(C)$. Given a vector of pure strategies, $\boldsymbol{P}=\{P(A), P(B), P(C)\}$, let $S(x|\boldsymbol{P})$ represent candidate x's share of the vote.

Proposition 1 *There exists a PSNE. In each equilibrium, either candidate A or candidate C receives a plurality of votes. Furthermore, P is an equilibrium in which candidate C wins if and only if:*

(a) $P(C) - P(A) < 2/3$,
(b) $3P(B) < 2 - P(C)$,
(c) $3P(C) > 2 - P(A)$.

In particular, if C wins, then $0 < P(A) \leq P(B) < 1/2 < P(C) < 1$.

Proof In the first two steps, we show that if $\boldsymbol{P}$ satisfies conditions (a) and (b), then C is the unique winner of the election. In the next three steps, we show that $\boldsymbol{P}$ is a PSNE under conditions (a)–(c).

Step 1. B cannot win. If $P(A) < P(B)$, then from condition (a), $S(B|\boldsymbol{P}) < 1/3$. When there are three candidates, the winner's share must be at least 1/3, therefore, B cannot win or tie the election. If $P(A) = P(B)$, then $S(A|\boldsymbol{P}) = S(B|\boldsymbol{P}) = (1/2)(P(B) + (1/2)[P(C) - P(B)]) < S(C|\boldsymbol{P}) = 1 - P(C) + (1/2)[P(C) - P(B)]$ as long as $3P(B) < 4 - 3P(C)$, which is implied by condition (b).

Step 2. A cannot win. If $P(A) = P(B)$, then A cannot win for the same reason as above. When $P(A) < P(B)$, $S(A|\boldsymbol{P}) = P(A) + (1/2)[P(B) - P(A)] = (1/2)[P(A) + P(B)]$. It is clear that A's share is maximized at $P(B) - \varepsilon$, for ε vanishingly small. Hence, $S(A|\boldsymbol{P}) < (1/2)[P(B) + P(B)] = P(B) < 1 - (1/2)$

$[P(C) + P(B)] = S(C|\boldsymbol{P})$ if $3P(B) < 2 - P(C)$ which is condition (b). Thus, C is the unique winner.

Step 3. No one deviates to a position greater than $P(C)$. If B deviates to $P(C) + \varepsilon$, then $S(B|\boldsymbol{P}') = 1 - P(C) - \varepsilon/2$; however, $S(A|\boldsymbol{P}')$ becomes $P(A) + (1/2)[P(C) - P(A)] > 1 - P(C)$ by condition (c) allowing A to win. If A deviates to $P(C) + \varepsilon$, then $S(B|\boldsymbol{P}') = (1/2)[P(C) + P(B)] > (1/2)[P(C) + P(A)]$ so that B would win. Hence, neither A nor B can win or tie the election by undertaking such a deviation.

Step 4. No one deviates to $P(C)$. If B deviates to $P(C)$, then, by condition (c), $S(A|\boldsymbol{P}') = P(A) + (1/2)[P(C) - P(A)] > S(B|\boldsymbol{P}') = (1/2)[1 - P(C) + (1/2)[P(C) - P(A)]]$. Moreover, as $P(A) \leq P(B)$, A also cannot win or tie by deviating to $P(C)$.

Step 5. No one deviates to a position less than $P(C)$. From step 1, we know that $A[B]$ will not deviate to a position between $P(B)$ and $P(C)$ [between $P(A)$ and $P(C)$] and that $A[B]$ will not deviate to $P(B)$ [$P(A)$]. From step 2, we know that A cannot win at their best position of $P(B) - \varepsilon$ and clearly they have no incentive to move further to the left. Furthermore, because $P(A) \leq P(B)$, a deviation by B to the left of $P(A)$ will yield a vote share for B that is smaller than A's previous losing vote share and, therefore, they will not make this deviation. Hence, any $\boldsymbol{P}$ that satisfies conditions (a)–(c) is a PSNE.

Step 6. We now show that for any PSNE satisfying conditions (a)–(c), it must be that $0 < P(A), P(B) < 1/2$, and $1/2 < P(C) < 1$. If $P(A) = 0$, then $P(C) < 2/3$ by condition (a), but then $3P(C) < 2 - P(A)$, which contradicts condition (c). Similarly, if $P(C) = 1$, then $P(A) > 1/3$ by condition (a), but condition (b) implies that $3P(B) < 1$ or, more specifically, that $P(B) < 1/3$, which contradicts $P(A) \leq P(B)$. Finally, if $P(C) \leq 1/2$, then condition (c) implies that $P(A) > 1/2$ which contradicts $P(A) \leq P(C)$. Therefore, $P(C) > 1/2$ which, from condition (b), yields that $P(B) < 1/2$. Hence, $0 < P(A) \leq P(B) < 1/2 < P(C) < 1$.

Step 7. We now show that if $\boldsymbol{P}$ is a PSNE where C is the unique winner and $0 < P(A) \leq P(B) < 1/2 < P(C) < 1$, then conditions (a)–(c) must hold. By the construction of the proof, it is evident that conditions (b) and (c) are necessary. Suppose that condition (a) is not satisfied so that $P(C) - P(A) > 2/3$. In this case, either B wins or ties the election, which contradicts C being the unique winner, or B can win by deviating to $4/3 - C$ which yields $S(B|\boldsymbol{P}') > 1/3 = S(C|\boldsymbol{P}')$. Hence, condition (a) is necessary.

To see that there are no PSNE in which B wins, suppose she does. Without loss of generality, if $P(B) \leq 1/2$, then a deviation by C to $P(B) + \varepsilon$, for ε is small enough, would allow C to win the election. □

An immediate implication of Proposition 1 is that, *in a three-candidate PSNE, the middle candidate can never win an election and the winner does not adopt the platform that is most preferred by the median voter.* This provides an important contrast to a two-candidate election whereby both candidates (and the eventual randomly chosen winner) locate at the median position. The addition of one more candidate moves all three candidates from the center, and the winner must adopt an extreme position. In addition, the allowable distance between the equilibrium platforms increases from 0 in the two-candidate case to almost 2/3 in the three-candidate case.

Proposition 1 also implies that the winner's position places bounds on the possible equilibrium platforms of the two losing candidates. Moreover, the amount by which the loser's platforms can differ is not monotonic with respect to the winning candidate's position. The closer the winner locates to 1/2 or 1, the closer the losers must locate to each other.

3 Pure strategy Nash equilibria in *N*-candidate elections

We define $\boldsymbol{P}_N$ as the set of N platforms. To construct PSNE in N-candidate elections when $N > 3$, we start by considering a three-candidate PSNE that satisfies Proposition 1, such as $\boldsymbol{P}_3 = \{0.25, 0.3, 0.7\}$. Now add two additional candidates to yield $\boldsymbol{P}_5 = \{0.25, 0.26, 0.28, 0.3, 0.7\}$. Although the rightmost candidate still has a plurality share, the candidate at 0.26 could win by deviating to 0.71. One way to preclude this type of deviation with N candidates is to require the candidate on the far right to be at least as extreme as the candidate on the far left. For example, $\boldsymbol{P}_5 = \{0.25, 0.26, 0.28, 0.3, 0.8\}$ precludes profitable deviations to the right of the winning candidate at 0.8. In this case, however, the candidate at 0.3 could win by moving to 0.76. This deviation does not increase the mover's share but it does reduce the previous winner's share by enough to make the deviation profitable. To preclude this type of deviation, we require that one-half the distance between the winner (on the far right) and the candidate third from the right (second closest to the winning candidate) is less than the share of the candidate on the far left. For example, $\boldsymbol{P}_5 = \{0.25, 0.26, 0.28, 0.3, 0.75\}$ satisfies this additional condition as $(1/2)(0.75 - 0.28) < 0.255$.

With the two conditions provided above as well as the condition that the candidate on the far left cannot win by moving closer to his closest neighbor and the condition that no other interior candidate has a winning share, we can provide a set of sufficient conditions for a PSNE in which the winner is on the far end of the political spectrum. In Proposition 2, these conditions are provided for a winner on the far right. A symmetric set exists for a winner on the far left. To facilitate the analysis, we label the candidates X_j, $j = \{1, 2, \ldots, N\}$, and order the candidates so that $P(X_1) \leq \cdots \leq P(X_N)$. Furthermore, to be able to consider the share of the leftmost candidate when more than one candidate adopts that position, we define α to be the number of candidates locating at $P(X_1)$. Hence, if three candidates locate at $P(X_1)$, then $\alpha = 3$, and $X_{\alpha+1} = X_4$ is the first candidate distinct from X_1. Finally, we denote β as the number of candidates locating at $P(X_{\alpha+1})$, so that candidate $X_{\alpha+\beta+1}$ is the candidate in the third leftmost position.

Proposition 2 $\boldsymbol{P}_N = \{P(X_j)\}$, $j = \{1, 2, \ldots, N\}$, *is a PSNE in an N-candidate election, whereby the unique plurality share winner is* X_N, *if the following conditions hold*:

(a) $1 - (1/2)[P(X_N) + P(X_{N-1})] > P(X_2)$;
(b) $1 - (1/2)[P(X_N) + P(X_{N-1})] > (1/2)[P(X_{j+1}) - P(X_{j-1})]$ *for every* $j = \{2, 3, \ldots, N-1\}$;

(c) $1 - P(X_N) < \frac{1}{2\alpha}[P(X_{\alpha+1}) + P(X_1)]$;
(d) *If* $\alpha = 1$, *then* $1 - P(X_N) < \frac{1}{2\beta}[P(X_{\alpha+\beta+1}) + P(X_{\alpha+\beta})]$;
(e) $(1/2)[P(X_N) - P(X_{N-2})] < \frac{1}{2\alpha}[P(X_{\alpha+1}) + P(X_1)]$.

Proof The proof is contained in Chisik and Lemke (2004).

One implication of Proposition 2 is that when $N > 3$, the losing candidates are not limited to positions on the same side of the spectrum. For example, $\boldsymbol{P}_5 = \{0.25, 0.35, 0.52, 0.6, 0.7\}$ is a PSNE and has more candidates on the winning side than there are on the losing side. A more interesting implication of Proposition 2 is that additional candidates in the region between $P(X_{\alpha+\beta})$ and $P(X_{N-1})$ have no effect on the set of equilibria. The above $\boldsymbol{P}_5$ continues to be a PSNE with the winner at 0.7 even if an infinite number of candidates are added at positions between 0.35 and 0.6. In fact, *given any PSNE that satisfies Proposition 2, it is possible to let N approach infinity without changing the general structure of equilibrium set.* This result occurs because these additional candidates have no effect on conditions (a), (c), and (d) and they introduce slack into conditions (b) and (e).

An additional policy relevant to the implication of the model is given by the alternative scenario whereby additional candidates do alter the equilibrium set. In particular, the PSNE configuration $\boldsymbol{P}_5 = \{0.15, 0.25, 0.5, 0.55, 0.85\}$ has a distance of 0.7 between the winner on the far right and the leftmost candidate. As a point of comparison, when $N = 3$, Proposition 1 requires this distance to be less than two-thirds and the more restrictive Proposition 2 implies a space of less than 6/11. More generally, let the N candidates locate at $\boldsymbol{P}_N = \{1/(N+2), 2/(N+2), \ldots, (N-1)/(N+2), (N+2/3)/(N+2)\}$. This configuration satisfies Proposition 2 and the winner locates at $P(X_N) = (N+2/3)/(N+2)$. As N approaches infinity, $P(X_N)$ approaches 1 and $P(X_1)$ approaches 0. Hence, *as the number of candidates approaches infinity, the space between the leftmost and rightmost candidates can approach unity in a PSNE whereby the winner adopts one of these extreme positions.*

We now consider other PSNE whereby the winner is not the rightmost (or leftmost) candidate so that condition (b) of Proposition 2 is not satisfied. For example, $\boldsymbol{P}_5 = \{0.18, 0.2, 0.5, 0.8, 0.82\}$ is a PSNE where the winner is at 0.5. Similarly, $\boldsymbol{P}_9 = \{0.08, 0.12, 0.25, 0.4, 0.5, 0.6, 0.75, 0.88, 0.92\}$ is a PSNE where the candidates at 0.25 and 0.75 are tied for the highest plurality share. These examples demonstrate that *in a PSNE when $N > 3$, there exists PSNE where the winner chooses the platform most preferred by the median voter, where multiple candidates obtain the highest plurality share and where the plurality share winner need not be the candidate with the leftmost or rightmost platform.*

With Proposition 1, we demonstrated that, for $N = 3$, there exists PSNE when the objective is winning the election that does not exist when the objective is vote maximization. When $N > 3$, the opposite can also occur. That is, a PSNE under vote maximization might not remain a PSNE when winning is the objective. For example, in a six-candidate election with vote maximization being the objective, $\boldsymbol{P}_6 = \{0.15, 0.15, 0.45, 0.55, 0.85, 0.85\}$ is a PSNE in which the third and fourth candidates tie the election. In this example, no candidate can unilaterally deviate and garner a greater share of the vote; however, either the third or the fourth candidate could unilaterally deviate to 0.5 and become the sole winner of the election. This counter-example establishes that *if N is large enough, then there* *exists PSNE under vote maximization that are not PSNE when each candidate's objective is to win the election.*

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