

Walking on Numbers and a Self-Referential Formula

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Coauthors for Walking on Numbers



Figure: Kevin Kupiec, Marina Rawlings and me. 

Background

- “Walking on Real Numbers” by Aragón, Bailey, Borwein and Borwein.
- Consider a number in base 4. For example π . In base 4,

$$\pi = 3.0210033312222020201122030020310301 \dots,$$

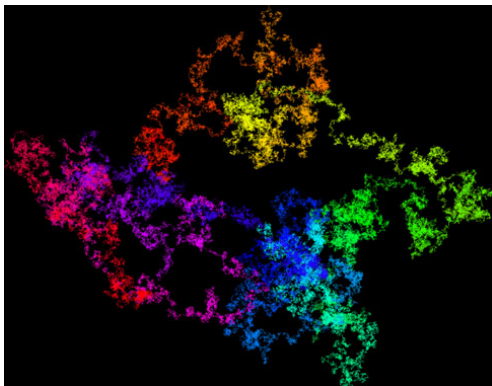
because

$$\pi = 3 + 0 \left(\frac{1}{4} \right) + 2 \left(\frac{1}{4^2} \right) + 1 \left(\frac{1}{4^3} \right) + \dots$$

- We start at the origin in the Cartesian plane. We move a unit to the right whenever we hit a digit 0, we move a unit up whenever we hit a digit 1, we move a unit left whenever we hit a digit 2 and we move a unit down if we hit a digit 3.

Walking on π

Walking on the first 100 billion digits of π reveals the following picture:



Walking on the number 419636198 which can be rewritten in base 4 as

Inspiration

10490122716774994374866192805654486016 17567358491560876166848380843144358447
 2528755516292470277595557045371567931 30587832477297720217708181879659063736
 57674879814228013285920278610192581409 57135748704712290267465151312805954195
 3997504202061380373822338959713391954

/

16122269626942909129404900662735492142 29880755725468512353395718465191353017
 34881431401750453996944547935301206438 33272670970079330526292030350920973600
 45095545613659664932507839146477284016 23856513742952945308961226815274887561
 5658076162410788075184599421938774835



Project

- For each letter of the alphabet, find a rational that satisfies that if you random walk through that rational, you get the letter of the alphabet.
- Build a computer program that can use the information above to figure out the rational that works for a particular phrase.

How do you find the rational for a particular letter?

- [illegible]

$$1 \left(\frac{1}{4} \right) + 0 \left(\frac{1}{4^2} \right) + 1 \left(\frac{1}{4^3} \right) + \dots =$$

6384779382043951036217348661253680515005885357484535471589654514956414794662721006368542
597248986985323127416704519810815261318970154183 /
2325883917745942049757836185241614509931652354199417792900768637378045721962873354643811
3622840434097944400691400517693873107252115668992

Alphabet Continued

One issue with just finding the rational as above, is that the random walk will now continue indefinitely to the right as the expansion ends with infinitely many zeroes. To fix this, we consider the number of digits in the representation of the letter and then do a geometric series expansion.

For example with the letter “D”, it has 227 digits. Let the rational representation be x . Then the rational that loops itself over and over would be

$$x + 4^{-227}x + \left(4^{-227}\right)^2 x + \dots = \frac{1}{1 - 4^{-227}}x.$$

Example: Numerator has 1905 digits

4705874961490126446537277294670458551293812882625937518122133640944156954742219078731712010388849418460127807197476666621733496509698279679954029487865846701455
 437930689539159852467848729686311441793195799757065894714215920012122357501181446750607856729376403286553855509883714417969030011403297239506484304528011197094
 827541555258276891886965965140436954440277651229749142662752767346321358905508951602662286266758823550172750364027080007324809172453760712720081986660529714140
 1428733038171010582302982079091534223471271779106076136022353633693617652156199225249695011514412036758804752808342596331299978944076425240876973303105294053877
 5697745685320784604707152715217569384095465636132402027911061192762153818164506514046245347505091620171701605178854363639892812998121473507451147489316937094150
 731772397210206296874054095030812798518677398657674496248018268272948558831079508708853233524495795997374513014203532980534324553989859801509301689518266191644
 0940332303654496896199283158180039041788710031461714548637825146245866418918713713335574587827964314963383798472125625341636774400366528709503666407655250326730
 30166820755810459463774230603105341225252573422767923341279340482458944472367610668767033913427820439856651268507159514593528543122001368820491267429437943183
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/
 8275849070206774095634522138903321027400076848280460847658750424098275279639947086184866739279124915285747856651764281967269344909268578667232896774903529354427
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 813713120965106282616930530218113684266960806212204298312234518225452466723390360245520551826283773694621281012291523793271211913523076795876645952637278933334
 0636907759461814294411227892174848354377456376708666007550929743082031178937027924179963909521061252909885245902228245975445641395874090274058551939533664831556
 616088122893136808243444742752210452864958870302620182384660309307504973338099862878262565715837560167111099062242118103279208117814101481270361978813425816344
 30176223164061005329576805734994506740026353824906561185946485733461848509871201070836061173483414989064198965861694015589601078463472025225521462331767074302
 4137889469485779163123909319517286920533617797836935157323303351520973386709824653290990533664676765126842626862419124823787305954163415524790443968783199084752
 41401068766633253562381611916644695958686797085339112409323586996376896908997926851371422292049054893908879929627873455028443700248862492645313069492329045033
 4213149660610574274390475661600179702214581584535818229575950709150352151869176283675683602241033147982452609253559170755365612225409567703368115422466718052783
 8058413859045796647997925758103975563649078373784383602092601414112843824678342058037033794867660715796001506879627533764919487144955516084043619228931668381298
 187841659167619733398058533694892732349514386481135476624010727576028920381445062513723325475282759883975931303694456188297528342981191360034375904126071835178
 29549503596811843689397695647797668844842659342944969209996478286058798447494723160616434608064972858999734236742359136131552149644811442323455.



Picture

AWESOME MATH

Suppose that for every letter α (a variable representing an uppercase letter from the English alphabet), you find a rational r_α and an integer n_α , such that the walk of r_α with n_α steps spells the letter α in such a way that the last step ends at the origin. We will also define $r_{blank} = n_{blank} = 0$, i.e., representing a blank space. Finally, suppose the base of each letter is at most w , i.e., the length of a blank space is w .

Theorem

Suppose we are given a sentence σ which we'll write as $\sigma = \alpha_1 \alpha_2 \alpha_3 \cdots \alpha_k$, where α_j is a letter or a space. Let

$$n = \sum_{i=1}^k n_{\alpha_i} + 2(k-1)w, \quad (1)$$

and

$$r = \sum_{i=1}^k \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1} (n_{\alpha_j} + w)\right)}} + \frac{2 \cdot 4^w \left(1 - \frac{1}{4^{(k-1)w}}\right)}{3 \cdot 4^{\left(\sum_{j=1}^k (n_{\alpha_j} + w)\right)}}. \quad (2)$$

Then a walk on r of length n spells out σ . Furthermore, a walk on $\frac{4^n}{4^n - 1}r$ of any length $m \geq n$ spells out σ .

$$\begin{aligned}
 r &= \sum_{i=1}^k \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1} (n_{\alpha_j} + w)\right)}} + \frac{r_{\alpha_{k+1}}}{4^{\left(\sum_{i=1}^k (n_{\alpha_i} + w) - w\right)}} \\
 &= \sum_{i=1}^k \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1} (n_{\alpha_j} + w)\right)}} + \frac{4^w}{4^{\left(\sum_{i=1}^k (n_{\alpha_i} + w)\right)}} \left(\frac{2}{4} + \frac{2}{4^2} + \frac{2}{4^3} + \dots + \frac{2}{4^{(k-1)w}} \right).
 \end{aligned}$$

Coauthor on a Self-Referential Formula



Figure: Margaret Fortman and me.

Tupper's self-referential formula

$$\frac{1}{2} < \left\lfloor \text{mod} \left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \text{mod}(\lfloor y \rfloor, 17)}, 2 \right) \right\rfloor$$

Graph of the above equation for $0 \leq x \leq 105$ and $k \leq y \leq k + 17$ is

$$\frac{1}{2} < \left\lfloor \text{mod} \left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \text{mod}(\lfloor y \rfloor, 17)}, 2 \right) \right\rfloor$$

for k equal to

4858450636189713423582095962494202044581400587983244549483093085061934704708809
 92845064476986552436484999724702491511911041160573917740785691975432657185544205721
 04457358836818298237541396343382251994521916512843483329051311931999535024137587652
 39264874613394906870130562295813219481113685339535565290850023875092856892694555974
 28154638651073004910672305893358605254409666435126534936364395712556569593681518433
 48576052669401612512669514215505395545191537854575257565907405401579290017659679654
 80064427829131488548259914721248506352686630476300

Project

For a given sentence, find the integer k such that the graph of Tupper's formula looks like that sentence for $0 \leq x \leq 105$ and $k \leq y \leq k + 17$.

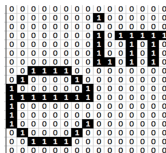
Example

AWESOME MATH SUMMER
CAMP MATH FORUM!
I ♥ MATHEMATICS

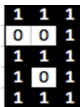
Plot of Tupper's formula for $0 \leq x \leq 105$ and $k \leq y \leq k + 17$
when k is

```
9448335586977921792094769338556771178300275962701467073136749206593612
0735852472992868069816358647582675186797278348788921505904934208907784
2231528202712792729556817081001305852775313239293311105710516185748280
4848480190236878024217073530490523670286117542560555168737661610834676
1916673090547662007717653594472190231943273090129737924404514048028593
0958629894589952336681323640961896549536555876946990062163900558734160
26665580028518227261673546391267368380541211252901353129766912
```

How to get it done



As the previous project, the key is figuring out how to do a letter first.



Letter a

1	1	1
0	0	1
1	1	1
1	0	1
1	1	1

The binary number for the letter a is 11101 10101 11111

Multiply by 2^{17} to move column to the right.

Formula for the lowercased a is:

$$17((1 + 2 + 4 + 16) + (1 + 4 + 16)2^{17} + (1 + 2 + 4 + 8 + 16)2^{34})$$

Theorem

Let $k = 17k'$ for a nonnegative integer $k' < 2^{106 \times 17}$. Suppose we write k' in binary as follows:

$$k' = \sum_{m=0}^{105} \sum_{n=0}^{16} a_{17m+n} 2^{17m+n}.$$

Then

$$\left[\text{mod} \left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \text{mod}(\lfloor y \rfloor, 17)}, 2 \right) \right] = a_b,$$

for $b = 17\lfloor x \rfloor + \text{mod}(\lfloor y \rfloor, 17)$.

Therefore, the point (x, y) is painted whenever $a_b = 1$ and not painted when $a_b = 0$, i.e., it depends only on the binary expansion of k' .

Theorem

Given a sentence $\sigma = \alpha_1 \alpha_2 \cdots \alpha_k$, where α_i represents a single letter or a blank space and $k \leq 63$, we use the following formula to figure out the value of k for the range where the plot of Tupper's formula is σ :

$$\sum_{i=1}^{\min(21,k)} 2^{85(i-1)+12} f(\alpha_i) + \sum_{i=22}^{\min(42,k)} 2^{85(i-22)+6} f(\alpha_i) + \sum_{i=43}^k 2^{85(i-43)} f(\alpha_i). \quad (3)$$

Proof.

Each letter fits in a block of width 5 and height 5. To move from one letter to the next (to the right), we need to multiply by $2^{17 \times 5} = 2^{85}$. This is where the 85's in the exponents come from. The reason we add 12 and 6 (depending on how many letters we have) is because the first row consists of numbers in the top "strip" ($k + 12 \leq y < y + 17$), so we have to multiply by 2^{12} to move upwards. The numbers in the middle strip ($k + 6 \leq y \leq y + 11$) need a shift of 2^6 , and the bottom row needs no translation. The formula follows. □

THANK YOU!

260449203849882664517035139638426870137917317601914921681571572073092
 8695649142399363187609280194294167878123410715564434829372484954369221
 6688674071161566295182034925025124500569335817787439872314183988703371
 664977034810026841939388184409144925414508598131377588818477056

THANK YOU!

244838419085865068867681259343425875183951488272013453075082431177592 0962050203904557003580654935217327434523590196726040466969501801090605
 566417280709734593212987599213260680646316630622078943630207743373850 7810762263639386835401550161949045105588754258631414471096220189442622
 1665427118990256510546429000993755547847309756192941247474274244110787 8660478187344039509845541294428633493975708789645581674249072107946910
 2765408044063942823929158994523762592434026978546624093591607103258197 445063872411745846773127551844055328481368483490252605636902712794432
 9883720681160278275882713626711046114853618093566074424105013908484306 7404052939064793858614545855382152074663980612417168491376157942144755
 1424213635574427532612628956112547281010460985326871870253127060031406 9020366919487757012852917356230205755317365507728854738429219183094869
 36959728218435989054297657583382856688534922378577986330045089333457 9166101811912895915031607121962558396750697854916244499605957383922242
 208776824286810034216057699362187013656283147052173522331783042174874 414009480411833840723288129692077185337544126726009078432030025686771
 52398094888054501646028761824177350795554860429732609262786222124676 600894826038486209391328014641316052644422022

38194793377395291670329838484906512027721021110801860682548053881724174 2920102672707614237693622767847895409616985207326646402685739366166208
 484062641421358736309283713598537082161795296341759840705122747193031 7627890537469797349469887395417031631058274040351838367832032124521106
 7194847614828901800156489387408054017648810457448634475529542970498905 7255432583059392620413461586291555620523049770255248031290472335172139
 241294421758805609256913422444425119749770655048540568111268913987049510 5866348693513857856730829398652779264556390283916331450215630875473236
 7905955243200201255079498650860927497492754285974565420665019701251 95912478551043203972465475081888151581997031018121793852983572059720856
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