Partitioning powers into sets of equal sum

Enrique Treviño
(joint work with Paul Pollack)

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Motivating Puzzle

Consider the following puzzle submitted by Dean Ballard to the Riddler column on the FiveThirtyEight website:

King Auric adored his most prized possession: a set of perfect spheres of solid gold. There was one of each size, with diameters of 1 centimeter, 2 centimeters, 3 centimeters, and so on. Their brilliant beauty brought joy to his heart. After many years, he felt the time had finally come to pass the golden spheres down to the next generation – his three children. He decided it was best to give each child precisely one-third of the total gold by weight, but he had a difficult time determining just how to do that. After some trial and error, he managed to divide his spheres into three groups of equal weight. He was further amused when he realized that his collection contained the minimum number of spheres needed for this division. How many golden spheres did King Auric have?

Translation and a variant

What is the smallest positive integer n such that $1^3, 2^3, \ldots, n^3$ can be partitioned into three sets with equal sum?

Question

For what positive integers n can the set $\{1^3, 2^3, \ldots, n^3\}$ be partitioned into three sets with equal sum?

Thinking it through

A partition of the set $\{n, n-1, n-2, n-3, n-4, n-5\}$ into three sets of equal sum is

$$\begin{array}{c|c|c} n & n-1 & n-2 \\ n-5 & n-4 & n-3 \end{array}$$

Note that

$$\begin{aligned} n^2 + (n-5)^2 &= 2n^2 - 10n + 25 \\ (n-1)^2 + (n-4)^2 &= 2n^2 - 10n + 17 \\ (n-2)^2 + (n-3)^2 &= 2n^2 - 10n + 13 \end{aligned}$$

Cycling to get a match

A	B	C
n	$n-1$	$n-2$
$n-5$	$n-4$	$n-3$
$n-8$	$n-6$	$n-7$
$n-9$	$n-11$	$n-10$
$n-13$	$n-14$	$n-12$
$n-16$	$n-15$	$n-17$

$$\sum_{a\in A} a^2 = 2n^2 - 10n + 25 + 2(n-6)^2 - 10(n-6) + 13 + 2(n-12)^2 - 10(n-12) + 17$$

$$\sum_{b\in B} b^2 = 2n^2 - 10n + 17 + 2(n-6)^2 - 10(n-6) + 25 + 2(n-12)^2 - 10(n-12) + 13$$

$$\sum_{c\in C} c^2 = 2n^2 - 10n + 13 + 2(n-6)^2 - 10(n-6) + 17 + 2(n-12)^2 - 10(n-12) + 25.$$

Nice Theorem

Theorem

Any set of 54 consecutive cubes can be partitioned into three sets of equal sum.

Generalization:

Theorem

Any $2m^k$ consecutive k-th powers can be partitioned into m sets of equal sum.

Note: This theorem was also proved by Choudhry (2020).

Connecting to the Math Literature

The Prouhet-Tarry-Escott (PTE) problem asks to find integers N, and a_{ij} for $i \in \{1, 2, \ldots, N\}, j \in \{1, 2, \ldots, m\}$ such that

$$\sum_{j=1}^{N} a_{1j} = \sum_{j=1}^{N} a_{2j} = \cdots = \sum_{j=1}^{N} a_{mj}$$
$$\sum_{j=1}^{N} a_{1j}^2 = \sum_{j=1}^{N} a_{2j}^2 = \cdots = \sum_{j=1}^{N} a_{mj}^2$$
$$\vdots$$
$$\sum_{j=1}^{N} a_{1j}^k = \sum_{j=1}^{N} a_{2j}^k = \cdots = \sum_{j=1}^{N} a_{mj}^k. \qquad (1)$$

The trivial PTE solutions are those for which there is an i and a $j \neq i$ for which the sets $\{a_{i1}, a_{i2}, \ldots, a_{iN}\}$, $\{a_{j1}, a_{j2}, \ldots, a_{jN}\}$ are the same.

What we know of PTE

Let $P(k,m)$ be the smallest positive integer N such that there is a nontrivial solution to PTE.

- $P(k,m) \geq k+1$
- (Borwein-Ingalls) $P(k,m) = k+1$ for $s \leq 10$ or $s = 12$ and $m = 2$, or for $k \in \{2,3,5\}$ for any m
- (Wright)

$$P(k,m) \leq \begin{cases} \frac{k^2+4}{2} & \text{for } m = 2 \\ \frac{k^2+3}{2} & \text{for odd } k \\ \frac{k^2+k+2}{2} & \text{otherwise} \end{cases}.$$

- Prouhet gave an explicit construction showing $P(k,m) \leq m^k$.

Our theorem in the PTE framework

Theorem

There is an explicit construction showing $P(k, m) \leq 2m^{k-1}$*.*

Back to our puzzle

- We need $1^3 + 2^3 + \ldots + n^3$ to be a multiple of 3. Therefore $n \equiv 0, 2 \mod 3$.
- We know that if it can be partitioned for n it can also be for $n + 54$.

Searching for partitions

n	Partition
23	$\{3,6,10,13,18,19,21\}, \{1,4,7,8,12,16,20,22\}, \{2,5,9,11,14,15,17,23\}$
26	$\{4,14,19,24,26\}, \{2,3,5,11,15,16,18,22,25\}, \{1,6,7,8,9,10,12,13,17,20,21,23\}$
27	$\{11,12,21,25,27\}, \{7,13,14,15,17,18,22,26\}, \{1,2,3,4,5,6,8,9,10,16,19,20,23,24\}$
29	$\{7,12,14,19,24,25,28\}, \{2,6,8,17,20,23,26,27\}, \{1,3,4,5,9,10,11,13,15,16,18,21,22,29\}$
30	$\{4,7,8,16,19,25,26,30\}, \{3,5,11,14,17,20,21,23,24,27\}, \{1,2,6,9,10,12,13,15,18,22,28,29\}$
32	$\{16,22,25,31,32\}, \{2,4,5,8,11,12,17,18,19,20,23,29,30\},$ $\{1,3,6,7,9,10,13,14,15,21,24,26,27,28\}$
33	$\{4,13,16,21,24,26,28,33\}, \{1,3,6,7,10,18,20,25,27,29,31\},$ $\{2,5,8,9,11,12,14,15,17,19,22,23,30,32\}$
35	$\{7,17,24,25,28,32,35\}, \{11,18,19,20,22,29,33,34\},$ $\{1,2,3,4,5,6,8,9,10,12,13,14,15,16,21,23,26,27,30,31\}$
36	$\{5,7,10,14,22,29,31,33,35\}, \{1,6,12,15,16,17,18,20,24,26,27,28,36\},$ $\{2,3,4,8,9,11,13,19,21,23,25,30,32,34\}$
38	$\{5,17,21,24,29,32,35,38\}, \{1,6,9,10,12,13,14,15,20,31,33,36,37\},$ $\{2,3,4,7,8,11,16,18,19,22,23,25,26,27,28,30,34\}$
39	$\{6,22,25,27,36,37,39\}, \{2,3,4,5,8,11,12,13,16,23,26,29,30,32,33,35\},$ $\{1,7,9,10,14,15,17,18,19,20,21,24,28,31,34,38\}$
41	$\{2,5,18,26,27,32,35,39,41\}, \{10,13,20,23,24,28,31,34,38,40\},$ $\{1,3,4,6,7,8,9,11,12,14,15,16,17,19,21,22,25,29,30,33,36,37\}$
42	$\{2,8,9,20,24,26,35,38,39,42\}, \{3,4,6,7,11,12,14,15,19,21,25,36,37,40,41\},$ $\{1,5,10,13,16,17,18,22,23,27,28,29,30,31,32,33,34\}$
44	$\{1,2,9,20,28,31,36,38,43,44\}, \{4,5,8,10,11,13,15,22,25,26,27,29,32,39,40,42\},$ $\{3,6,7,12,14,16,17,18,19,21,23,24,30,33,34,35,37,41\}$
45	$\{5,10,11,19,24,27,28,29,32,34,36,40,44\}, \{4,9,14,15,22,23,26,30,31,37,39,41,42\},$ $\{1,2,3,6,7,8,12,13,16,17,18,20,21,25,33,35,38,43,45\}$
47	$\{16,24,25,28,34,38,43,45,47\}, \{5,7,10,20,21,31,35,36,37,40,42,46\},$ $\{1,2,3,4,6,8,9,11,12,13,14,15,17,18,19,22,23,26,27,29,30,32,33,39,41,44\}$
48	$\{8,10,12,19,22,24,25,27,32,36,37,39,45,48\}, \{2,3,5,6,9,11,18,20,23,29,40,41,42,46,47\},$ $\{1,4,7,13,14,15,16,17,21,26,28,30,31,33,34,35,38,43,44\}$

Searching for partitions 2

n	Partition
50	{6, 12, 16, 28, 33, 36, 41, 42, 43, 45, 49}, {1, 2, 9, 18, 19, 20, 22, 25, 27, 29, 30, 31, 34, 35, 37, 39, 46, 47}, {3, 4, 5, 7, 8, 10, 11, 13, 14, 15, 17, 21, 23, 24, 26, 32, 38, 40, 44, 48, 50}
51	{2, 8, 11, 15, 20, 26, 32, 37, 42, 44, 45, 47, 49}, {3, 14, 16, 22, 25, 28, 30, 31, 34, 35, 38, 43, 50, 51}, {1, 4, 5, 6, 7, 9, 10, 12, 13, 17, 18, 19, 21, 23, 24, 27, 29, 33, 36, 39, 40, 41, 46, 48}
53	{4, 6, 13, 17, 18, 21, 30, 32, 46, 47, 49, 51, 53}, {3, 8, 24, 25, 27, 31, 36, 38, 42, 44, 45, 48, 52}, {1, 2, 5, 7, 9, 10, 11, 12, 14, 15, 16, 19, 20, 22, 23, 26, 28, 29, 33, 34, 35, 37, 39, 40, 41, 43, 50}
54	{17, 22, 38, 39, 47, 48, 49, 51, 52}, {4, 5, 18, 24, 26, 33, 36, 40, 42, 43, 45, 53, 54}, {1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 20, 21, 23, 25, 27, 28, 29, 30, 31, 32, 34, 35, 37, 41, 44, 46, 50}
56	{3, 5, 8, 11, 16, 23, 26, 27, 30, 31, 32, 38, 43, 44, 45, 47, 52, 53}, {2, 6, 7, 9, 10, 12, 13, 17, 18, 19, 20, 21, 22, 24, 25, 29, 33, 35, 40, 41, 46, 48, 54, 55}
57	{7, 31, 40, 52, 53, 55, 56, 57}, {2, 3, 12, 13, 17, 18, 20, 22, 23, 25, 26, 27, 29, 32, 36, 38, 39, 41, 42, 43, 47, 48, 54}, {1, 4, 5, 6, 8, 9, 10, 11, 14, 15, 16, 19, 21, 24, 28, 30, 33, 34, 35, 37, 44, 45, 46, 49, 50, 51}
59	{1, 2, 6, 9, 10, 18, 23, 27, 37, 38, 43, 46, 55, 56, 57, 58}, {3, 4, 5, 11, 13, 15, 17, 20, 30, 33, 34, 35, 36, 44, 48, 50, 53, 54, 59}, {7, 8, 12, 14, 16, 19, 21, 22, 24, 25, 26, 28, 29, 31, 32, 39, 40, 41, 42, 45, 47, 49, 51, 52}
60	{14, 16, 23, 26, 30, 31, 32, 35, 38, 40, 42, 43, 44, 47, 50, 56, 57}, {1, 2, 3, 5, 7, 11, 13, 15, 19, 21, 29, 33, 36, 37, 45, 49, 53, 55, 58, 60}, {4, 6, 8, 9, 10, 12, 17, 18, 20, 22, 24, 25, 27, 28, 34, 39, 41, 46, 48, 51, 52, 54, 59}
62	{2, 4, 7, 9, 11, 21, 30, 31, 48, 50, 51, 57, 58, 60, 62}, {1, 3, 8, 10, 15, 18, 19, 22, 29, 33, 39, 42, 47, 49, 52, 53, 54, 56, 59}, {5, 6, 12, 13, 14, 16, 17, 20, 23, 24, 25, 26, 27, 28, 32, 34, 35, 36, 37, 38, 40, 41, 43, 44, 45, 46, 55, 61}
63	{1, 4, 5, 8, 15, 23, 24, 29, 35, 42, 46, 51, 53, 55, 58, 60, 61}, {7, 10, 11, 12, 14, 17, 18, 21, 28, 33, 34, 37, 38, 39, 40, 44, 47, 48, 49, 52, 59, 62}, {2, 3, 6, 9, 13, 16, 19, 20, 22, 25, 26, 27, 30, 31, 32, 36, 41, 43, 45, 50, 54, 56, 57, 63}

Searching for partitions 3

n	Partition
65	$\{1, 7, 8, 18, 22, 25, 27, 32, 38, 40, 48, 50, 52, 55, 63, 64, 65\}$ $\{2, 4, 6, 11, 14, 15, 16, 19, 21, 28, 30, 34, 37, 39, 41, 43, 49, 56, 58, 59, 61, 62\}$, $\{3, 5, 9, 10, 12, 13, 17, 20, 23, 24, 26, 29, 31, 33, 35, 36, 42, 44, 45, 46, 47, 51, 53, 54, 57, 60\}$
66	$\{3, 7, 9, 13, 17, 18, 23, 31, 34, 41, 46, 49, 53, 55, 58, 61, 62, 65\}$, $\{2, 4, 6, 8, 14, 20, 24, 29, 32, 36, 37, 38, 40, 47, 48, 54, 59, 60, 63, 66\}$, $\{1, 5, 10, 11, 12, 15, 16, 19, 21, 22, 25, 26, 27, 28, 30, 33, 35, 39, 42, 43, 44, 45, 50, 51, 52, 56, 57, 64\}$
68	$\{1, 2, 6, 7, 13, 33, 36, 37, 41, 48, 54, 55, 57, 58, 62, 64, 68\}$, $\{3, 5, 11, 16, 17, 20, 22, 23, 25, 26, 27, 32, 38, 42, 47, 49, 50, 51, 52, 59, 60, 63, 66\}$, $\{4, 8, 9, 10, 12, 14, 15, 18, 19, 21, 24, 28, 29, 30, 31, 34, 35, 39, 40, 43, 44, 45, 46, 53, 56, 61, 65, 67\}$
69	$\{1, 6, 7, 28, 33, 38, 39, 40, 55, 57, 60, 63, 65, 67, 68\}$, $\{3, 4, 9, 14, 15, 20, 21, 22, 24, 25, 30, 31, 32, 35, 45, 49, 58, 61, 62, 64, 66, 69\}$, $\{2, 5, 8, 10, 11, 12, 13, 16, 17, 18, 19, 23, 26, 27, 29, 34, 36, 37, 41, 42, 43, 44, 46,$ $47, 48, 50, 51, 52, 53, 54, 56, 59\}$
71	$\{2, 8, 14, 18, 20, 22, 31, 35, 37, 39, 43, 46, 47, 51, 53, 55, 57, 61, 62, 66, 67\}$, $\{4, 5, 9, 10, 11, 13, 17, 24, 26, 28, 30, 33, 38, 41, 44, 49, 52, 56, 58, 63, 64, 70, 71\}$, $\{1, 3, 6, 7, 12, 15, 16, 19, 21, 23, 25, 27, 29, 32, 34, 36, 40, 42, 45, 48, 50, 54, 59, 60, 65, 68, 69\}$
72	$\{1, 2, 3, 5, 10, 17, 25, 26, 34, 38, 46, 49, 52, 54, 55, 56, 59, 61, 62, 68, 69\}$, $\{4, 7, 12, 14, 15, 20, 23, 24, 28, 29, 33, 40, 41, 48, 50, 51, 53, 57, 58, 60, 66, 67, 70\}$, $\{6, 8, 9, 11, 13, 16, 18, 19, 21, 22, 27, 30, 31, 32, 35, 36, 37, 39, 42, 43, 44, 45, 47, 63, 64, 65, 71, 72\}$
74	$\{1, 2, 4, 8, 9, 18, 21, 22, 27, 29, 46, 47, 50, 52, 56, 57, 64, 67, 70, 72, 73\}$, $\{5, 6, 10, 11, 12, 13, 14, 17, 32, 33, 34, 43, 44, 48, 49, 51, 53, 62, 63, 65, 66, 68, 74\}$, $\{3, 7, 15, 16, 19, 20, 23, 24, 25, 26, 28, 30, 31, 35, 36, 37, 38, 39, 40, 41, 42, 45, 54, 55, 58, 59, 60, 61, 69, 71\}$
75	$\{1, 2, 3, 6, 7, 12, 13, 17, 25, 26, 40, 43, 48, 57, 59, 60, 64, 71, 72, 73, 75\}$, $\{4, 9, 19, 20, 23, 24, 28, 34, 37, 38, 46, 47, 49, 50, 51, 54, 55, 56, 61, 63, 65, 69, 70\}$, $\{5, 8, 10, 11, 14, 15, 16, 18, 21, 22, 27, 29, 30, 31, 32, 33, 35, 36, 39, 41, 42, 44, 45, 52, 53, 58, 62, 66, 67, 68, 74\}$
78	$\{1, 6, 11, 12, 20, 31, 40, 42, 43, 45, 46, 54, 62, 67, 69, 72, 74, 76, 78\}$, $\{2, 3, 4, 5, 7, 9, 22, 33, 37, 38, 48, 49, 51, 52, 55, 59, 63, 64, 71, 73, 75, 77\}$, $\{8, 10, 13, 14, 15, 16, 17, 18, 19, 21, 23, 24, 25, 26, 27, 28, 29, 30, 32, 34, 35, 36, 39, 41, 44, 47,$ $50, 53, 56, 57, 58, 60, 61, 65, 66, 68, 70\}$

Answer to the Puzzle and to some variants

Theorem

The $\{1^3, 2^3, \dots, n^3\}$ *can be partitioned into three sets of equal sum if and only if* $n = 23$ *or* $n \geq 26$ *with* $n \equiv 0, 2 \mod 3$.

Theorem

1. *The first n cubes can be partitioned into two sets of equal sum if and only if* $n \geq 12$ *and* $n \equiv 0, 3 \mod 4$.
2. *The first n squares can be partitioned into two sets of equal sum if and only if* $n \geq 7$ *and* $n \equiv 0, 3 \mod 4$.
3. *The first n squares can be partitioned into three sets of equal sum if and only if* $n \geq 18$ *and* $n \equiv 0, 4, 8 \mod 9$.

The classification problem in the PTE setting

Let L_m be the set of n's for which there exist two disjoint sets A, B of length n that satisfy that the sum of their j-th powers are equal for $0 \leq j \leq m-1$.

JOE BUHLER, SHAHAR GOLAN, ROB PRATT, AND STAN WAGON

m	L_m
1	$2+2\,\mathbb{N}$
2	$4+4\,\mathbb{N}$
3	$8+4\,\mathbb{N}$
4	$16+8\,\mathbb{N}$
5	$32+8\,\mathbb{N}$
6	$\{48\} \cup (64+8\,\mathbb{N})$
7	$\{96, 112, 128, 144, 160, 176\} \cup (192+8\,\mathbb{N})$
8	$\{144\} \cup (192+16\,\mathbb{N})$

TABLE 1. L_m, for $m \leq 8$.

Figure: The values for $m \leq 6$ on the table were proved by Boyd, Berend and Golan.

Classification problem for partitioning powers

We want to classify, given k, for what n can we partition the set $S_{n,k} = \{1^k, 2^k, \ldots, n^k\}$ into m sets with equal sum.

1. $S_{n,2}$ can be partitioned into 2 sets of equal sum if and only if $n \geq 7$ and $n \equiv 0, 3 \mod 4$.
2. $S_{n,2}$ can be partitioned into 3 sets of equal sum if and only if $n \geq 18$ and $n \equiv 0, 4, 8 \mod 9$.
3. $S_{n,3}$ can be partitioned into 2 sets of equal sum if and only if $n \geq 12$ and $n \equiv 0, 3 \mod 4$.
4. $S_{n,3}$ can be partitioned into 3 sets of equal sum if and only if $n = 23$ or $n \geq 26$ with $n \equiv 0, 2 \mod 3$.

Noticing a Pattern

1. $$\frac{1^2+2^2+\cdots+n^2}{2} \in \mathbb{Z} \text{ if and only if } n \equiv 0, 3 \bmod 4.$$
2. $$\frac{1^2+2^2+\cdots+n^2}{3} \in \mathbb{Z} \text{ if and only if } n \equiv 0, 4, 8 \bmod 9.$$
3. $$\frac{1^3+2^3+\cdots+n^3}{2} \in \mathbb{Z} \text{ if and only if } n \equiv 0, 3 \bmod 4.$$
4. $$\frac{1^3+2^3+\cdots+n^3}{3} \in \mathbb{Z} \text{ if and only if } n \equiv 0, 2 \bmod 3.$$

Theorem for large enough n

Theorem (Pollack-Treviño)

Let k and m be positive integers. There exists a constant C such that if $n \geq C = C(k, m)$ and $(1^k + 2^k + \cdots + n^k)/m$ is an integer, then $\{1^k, 2^k, \ldots, n^k\}$ can be partitioned into m sets of equal sum.

For example,

$$C(2,2) = 7,$$
$$C(2,3) = 18,$$
$$C(3,2) = 12,$$
$$C(3,3) = 26.$$

Detour - Waring's Problem

Let $g(k)$ be the least integer s for which any positive integer n can be written as a sum of s k-th powers. For example $g(2) = 4$ because any $n \equiv 7 \mod 8$ cannot be written as a sum of three squares, and

Theorem (Lagrange)

Every positive integer n can be written as a sum of four squares.

Hilbert in 1909 proved $g(k)$ exists for all k. It is conjectured that

$$g(k) = 2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2.$$

Waring's problem refinements

Theorem

For n large, there are many representations of n as a sum of s k-th powers, including many with s distinct k-th powers.

Theorem (Wright (1948))

For each fixed positive integer k there is an $s_0 = s_0(k)$ for which the following holds. Fix a positive integer $s \geq s_0$, and fix positive real numbers $\lambda_1, \ldots, \lambda_s$ with $\lambda_1 + \cdots + \lambda_s = 1$. If n is sufficiently large, there are positive integers $m_1, \ldots, m_s$ with

$$m_1^k + \cdots + m_s^k = n.$$

Furthermore, one can choose $m_1, \ldots, m_s$ such that each $m_i^k = (\lambda_i + o(1))n$, as $n \to \infty$.

Classification problem for two sets

- Suppose $T = \frac{1^k+2^k+\cdots+n^k}{2}$ is an integer. Let n be very large. We want to find a subset A of $\{1^k, 2^k, \ldots, n^k\}$ such that the sum of the k-th powers of A is T.
- Approach with a greedy algorithm, i.e., include the highest powers you can. Find the largest r such that

$$n^k + (n-1)^k + \ldots + (n-r)^k < T.$$

- Idea: Fill the gap using Waring's problem.
- Let B be the complement of A in $\{1^k, 2^k, \ldots, n^k\}$.

Obstacle in extending this idea to more than two sets

- Suppose you want three sets. Let $T = \frac{1^k+2^k+\cdots+n^k}{3}$.
- You use the greedy strategy from before to find $A \subseteq \{1^k, 2^k, \ldots, n^k\}$ such that

$$\sum_{a \in A} a = T.$$

- You use a greedy strategy with what's left to find $B \subseteq \{1^k, 2^k, \ldots, n^k\}$ such that

$$\sum_{b \in B} b = T.$$

- How do you confirm that A and B are disjoint?

Key corollary to overcome the obstacle

Corollary

For each fixed positive integer k there is an $s_0 = s_0(k)$ for which the following holds. Fix a positive integer $s \geq s_0$. For all large enough n, there are distinct positive integers $m_1, \ldots, m_s$ with $m_1^k + \cdots + m_s^k = n$ and each $m_i^k \in (\frac{n}{2s}, \frac{3n}{2s})$.

Proof.

This follows immediately from Wright's Theorem, choosing $\lambda_1, \ldots, \lambda_s$ as s distinct real numbers that sum to 1 from the interval $(\frac{1}{2s}, \frac{3}{2s})$. □

Sketch of Proof of Classification Theorem for large n

- Let $T = \frac{1^k+2^k+\cdots+n^k}{m}$.
- Let r_1 be the largest positive integer such that $n^k + (n-1)^k + \cdots + (n-r_1)^k < T$. Then include $n^k, (n-1)^k, \ldots, (n-r_1+1)^k$ in A_1.
- Use small powers in a certain range (via the Corollary) to fill in the gap to create A_1 such that $\sum_{a \in A_1} a = T$.
- Find the largest r_2 such that $(n-r_1)^2 + (n-r_1-1)^2 + \cdots + (n-r_1-r_2)^2 < T$. Include $(n-r_1)^k, (n-r_1-1)^k, \ldots, (n-r_1-r_2+1)^k$ in A_2. Use small powers in a certain range to fill in the gap to create A_2 such that $\sum_{a \in A_2} a = T$.
- Repeat the process to get $A_3, A_4, \ldots, A_{m-1}$.
- Let A_m be the rest of the powers.

Thank you

Thank You