Nice Bijective Proof

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Theorem 1. *Let $\tau(n)$ be the number of divisors of n. Let $\omega(n)$ be the number of distinct prime factors of n. Then*

$$\tau(n^2) = \sum_{d \mid n} 2^{\omega(d)}.$$

Proof. Let

$$T_n = \{(x, y) \in \mathbb{N}^2 \mid xy = n^2\} \text{ and } \Omega_d = \{(x, y) \in \mathbb{N}^2 \mid xy = d \text{ and } \gcd(x, y) = 1\}.$$

Then $|T_n| = \tau(n^2)$ and $|\Omega_d| = 2^{\omega(d)}$. Now, let

$$\Omega = \bigcup_{d \mid n} \Omega_d.$$

Then $|\Omega| = \sum_{d \mid n} 2^{\omega(d)}$. Therefore what we want to prove is that $|T_n| = |\Omega|$, which we will prove with a bijection.

Let $f : T_n \to \mathbb{N}^2$ be defined by

$$f(x, y) = \left(\frac{x + n}{\gcd(x + n, y + n)}, \frac{y + n}{\gcd(x + n, y + n)} \right).$$

We will show that the image of f is contained in Ω. Therefore $f : T_n \to \Omega$.

Let's prove that $f(T_n) \subseteq \Omega$. If $(x, y) \in T_n$, then $xy = n^2$. Then $(x+n)(y+n) = 2n^2+n(x+y) = n(2n+x+y) = n(x+n+y+n)$. Let $d = \gcd(x + n, y + n)$ and $x + n = dx_1, y + n = y_1$. Then

$$x_1 y_1 = \left(\frac{x + n}{d}\right)\left(\frac{y + n}{d}\right) = \left(\frac{n}{d}\right)\left(\frac{x + n + y + n}{d}\right) = \left(\frac{n}{d}\right)(x_1 + y_1).$$

But $\gcd(x_1 y_1, x_1 + y_1) = 1$, therefore $x_1 y_1 \mid n$ and $(x_1 + y_1) \mid d$. In particular $x_1 y_1 \mid n$ implies $(x_1, y_1) \in \Omega$ since $\gcd(x_1, y_1) = 1$, which is what we wanted to prove.

Now, let $g : \Omega \to \mathbb{N}^2$ be defined by

$$g(x,y) = \left(\frac{nx}{y}, \frac{ny}{x}\right).$$

It is easy to see that $g(\Omega) \subseteq T_n$ since $\left(\frac{nx}{y}\right)\left(\frac{ny}{x}\right) = n^2$. Hence $g : \Omega \to T_n$.

To complete the proof we need only show that $f \circ g$ and $g \circ f$ are the identity.

Let $(x,y) \in \Omega$. Then $g(x,y) = \left(\frac{nx}{y}, \frac{ny}{x}\right)$ and $f \circ g(x,y) = f\left(\frac{nx}{y}, \frac{ny}{x}\right)$. To calculate f we need to find $d = \gcd\left(\frac{nx}{y} + n, \frac{ny}{x} + n\right)$. Now $xy\,d = \gcd\left((nx+ny)x, (nx+ny)y\right) = nx + ny$ since $\gcd(x,y) = 1$ because $(x,y) \in \Omega$. Therefore $d = \frac{nx+ny}{xy}$. And we can now calculate $f \circ g(x,y)$.

$$f \circ g(x,y) = f\left(\frac{nx}{y}, \frac{ny}{x}\right) = \left(\frac{\frac{nx}{y} + n}{d}, \frac{\frac{ny}{x} + n}{d}\right) = \left(\frac{\frac{nx+ny}{y}}{\frac{nx+ny}{xy}}, \frac{\frac{nx+ny}{x}}{\frac{nx+ny}{xy}}\right) = (x,y).$$

Now, let $(x,y) \in T_n$. Let $d = \gcd(x+n, y+n)$. Then we have

$$g \circ f(x,y) = g\left(\frac{x+n}{d}, \frac{y+n}{d}\right) = \left(n\left(\frac{x+n}{y+n}\right), n\left(\frac{n+y}{n+x}\right)\right) = (x,y).$$

The last equality comes from the fact that $xy = n^2$, therefore $n\left(\frac{x+n}{y+n}\right) = \frac{nx+n^2}{n+y} = \frac{nx+ny}{n+y} = \frac{x(n+y)}{n+y} = x$. Similarly for the second coordinate.

Given that $f \circ g$ and $g \circ f$ are each the identity, we have a bijection proving that $|T_n| = |\Omega|$. $\square$