Abstract

From our early years of education we learn that polynomials can be factored to find their roots. In 1797 Gauss proved the Fundamental Theorem of Algebra, which states that every polynomial every polynomial can be factored into quadratic and linear products. Here we build up the necessary background in advanced complex analysis to prove a variant of the Fundamental Theorem of Algebra, namely that every polynomial has at least one complex root. The proof we show here uses Cauchy's Integral Formula and Liouville's Theorem, which we develop and prove. This leads us into the brilliant ideas of conforming complex maps into each other and the limits we can push complex functions to.

To the Math and Computer Science department for fostering
my curiosity;
To Professor Treviño for his guidance and patience;
To those who offered their support;
Thank you.

1 Introduction

The **roots**, or zeros, of a polynomial $P(x)$, are the solutions to the equation $P(x) = 0$. In simpler terms, it consists of the values of x that make the polynomial equal to 0. The Fundamental Theorem of Algebra states that every non-constant polynomial with complex coefficients has at least one complex root. Since real numbers are complex numbers with the imaginary part equal to zero, this includes all real polynomials too.

In the year 1797, a mathematician named Carl Friedrich Gauss published something very similar to The Fundamental Theorem of Algebra. Gauss showed that every polynomial $P(x)$ with real coefficients can be factored into linear and quadratic factors which is shown in [3]. He was unsatisfied with his proof and actually proved the same theorem three more times in his lifetime. Before then, every proof given of the Fundamental Theorem of Algebra relied on the polynomial having complex roots, and to Gauss these were deficient. Current proofs were based on the assumption that complex roots existed, and Gauss thought you must first prove their existence for this to be valid. Thus he only used algebra with a touch of geometry and analysis in his multiple proofs. Still there was something missing, and it was not discovered until the beginning of the 19th century.

This paper will develop the necessary steps to proving the modern version of the proof of The Fundamental Theorem of Algebra. Section 2 will give a brief introduction to complex numbers, and bring over vital concepts from the real numbers like limits and continuity. In section 3 derivatives and the concept of analyticity will be introduced, followed by a proof of the necessary Cauchy-Riemann equations. Section 4 builds upon everything stated so far and then delves into complex integration, which has a complexity level befitting of its title. Section 4 goes through contours, and different deformations, and definitions that can be applied to understand taking a integral the same. We end the section with Cauchy's Integral Theorem which will play a key role in the final proof. In section 5, we finally prove that every polynomial has at least one complex root. First Liouville's

Theorem is proved as a necessary lemma, then this is used with other proved theorems to show that the Fundamental Theorem of Algebra holds. Section 6 will move on to discuss the importance of this theorem and a few applications.

2 Complex Numbers

Before we get into the denser complex analysis required to prove The Fundamental Theorem of Algebra, an underlying understanding of complex numbers is needed.

2.1 Definitions

Definition 2.1. *A **complex number** is an expression of the form $a + bi$, where a and b are real numbers. Two complex numbers $a+bi$ and $c+di$ are said to be equal ($a + bi = c + di$) if and only if $a = c$ and $b = d$. We can also define $|z| = \sqrt{a^2 + b^2}$.*

The definition of an open set is the same as in real analysis,

Definition 2.2. *A set is S **open** in $\mathbb{C}$ if for all $z_0 \in S : \exists \epsilon > 0 \in \mathbb{R}$ such that $|z - z_0| < \epsilon \therefore z \in S$.*

Now that we have defined how a complex number looks and can be represented in terms of real numbers, our text step is to translate convergence over the complex numbers. Using the same definition of convergence from reals, take an infinite sequence $z_1, z_2, z_3, \ldots$ of complex numbers. We say that the complex number z_0 is the limit of the sequence (or converges to) if, for n large enough, z_n is close to z_0. An exact definition is

Definition 2.3. *A sequence of complex numbers $\{z_n\}_1^\infty$ is said to have the **limit** z_0 or to **converge** to z_0, and we write*

$$\lim_{n\to\infty} z_n = z_0$$

if for any $\epsilon > 0$ there exists an integer N such that $|z_n - z_0| < \epsilon$ for all $n > N$.

Graphically this means there is an open disk of radius ϵ centered at z_0 where all points z_n are inside the disk whenever $n > N$. Similarly, the concept of a complex valued function $f(z)$ having a limit is brought over. We give the definition,

Definition 2.4. *Let f be a function defined in some neighborhood of z_0, with the possible exception of the point z_0 itself. We say that the limit of $f(z)$ as z approaches z_0 is the number w_0 and write*

$$\lim_{z \to z_0} f(z) = w_0$$

if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

We can also show this graphically by stating that any neighborhood of $f(z_0)$ contains all the points that are contained in some neighborhood of z_0. From this you can see that a function f will be continuous at z_0 if the limit at the point exists and it the limit equals $f(z_0)$. So we define continuity as

Definition 2.5. *Let f be a function defined in a neighborhood of z_0. Then f is continuous at z_0 if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

An important rule we will need for later is the limit and furthermore the continuity of rational functions. First we will show the limit of a rational function exists.

Lemma 2.1. *Let g be continuous at z_0 with $g(z_0) \neq 0$. Then there are a $\delta > 0$ and $\alpha > 0$ such that if $|z - z_0| < \delta$, then $|g(z)| \geq \alpha$.*

Proof: Let

$$\alpha = \frac{|g(z_0)|}{2} > 0.$$

There is a $\delta > 0$ such that for $|z - z_0| < \delta$ we have

$$|g(z) - g(z_0)| \leq \alpha.$$

Therefore,

$$|g(z)| \geq |g(z_0)| - |g(z) - g(z_0)| \geq |g(z_0)| - \alpha = \frac{|g(z_0)|}{2} = \alpha.$$

QED

Theorem 2.2. *If* $\lim_{z\to z_0} f(z) = A$ *and* $\lim_{z\to z_0} g(z) = B$, *then*

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{A}{B} \text{ if } B \neq 0.$$

Furthermore if $f(z)$ *and* $g(z)$ *are continuous at* z_0, *then the quotient* $f(z)/g(z)$ *is also continuous at* z_0 *provided* $g(z_0) \neq 0$

Proof: We know from Lemma 2.1 that $g(z)$ has a lower bound, in some neighborhood, α. Since $f(z)$ is continuous, there is a δ_1 ¿ 0 such that whenever

$$0 < |z - z_0| < \delta_1 \implies |f(z) - A| < \frac{\epsilon\alpha B}{2B}.$$

Also since $g(z)$ is continuous there is a δ_2 such that whenever

$$0 < |z - z_0| < \delta_2 \implies |g(z) - B| < \frac{\epsilon\alpha B}{2A}.$$

Now let $\delta = \min\{\delta_1, \delta_2\}$. We want to show that

$$\left|\frac{f(z)}{g(z)} - \frac{A}{B}\right| < \epsilon.$$

If we manipulate the left side of this equation it becomes

$$\frac{1}{|g(z)B|}|f(z)B - g(z)A|.$$

We can add and subtract the term AB and then factor like terms out like so,

$$\frac{1}{|g(z)B|}|f(z)B + AB - AB - g(z)A| = \frac{1}{|g(z)B|}\left(|B||f(z) - A| + |A||g(z) - B|\right).$$

Since we know both f and g are continuous, we know this becomes

$$\frac{1}{|g(z)B|}(|B|\frac{\epsilon\alpha B}{2B} + |A|\frac{\epsilon\alpha B}{2A}) < \frac{1}{|g(z)B|} \times \epsilon\alpha B < \epsilon.$$

This satisfies the definition of continuity but only works if $g(z_0) \neq 0$.

QED

This fact lets us prove another theorem used later in the paper.

Theorem 2.3. *For a polynomial* $P(z)$ *where the limit*

$$\lim_{z\to z_0} P(z) = P(z_0)$$

exists for all z_0, *the rational* $\frac{1}{P(z)}$ *is continuous everywhere except when* $P(z) = 0$.

Proof: We want to take the limit of the function $\frac{1}{P(z)}$

$$\lim_{z \to z_0} \frac{1}{P(z)}.$$

Then if we apply the limit rule proved in the last theorem, we get

$$\frac{\lim_{z \to z_0} 1}{\lim_{z \to z_0} P(z)} = \frac{1}{P(z_0)}.$$

The limit on the denominator exists for every value of z so the only time this fails is when the denominator is 0. Hence, the only time this function is not continuous is when $P(z) = 0$. QED

Some of the important properties from the reals can be brought over to the complex numbers and now that we can check limits and continuity this will help us in the future to keep understanding the complex realm.

3 Analytic Functions

Now that we have defined limits and continuity, we can introduce a vital topic used in complex analysis, the analytic function. Before we can define this, we need to understand how to take derivatives in complex functions. The idea is simple, we have to split a complex function $f(z)$ into $u(x, y)$ and $v(x, y)$, its real and imaginary parts respectively. Then take the derivative as you would with real functions. The following definition extends from the real case.

3.1 Definitions

Definition 3.1. *Let f be a complex-valued function defined in a neighborhood of z_0. Then the derivative of f at z_0 is given by*

$$\frac{df}{dz}(z_0) \equiv f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{1}$$

provided this limit exists.

The only tricky part with this definition is that Δz can approach 0 in infinitely many ways since a complex number is on a plane. It is also easy to prove that many of the basic rules of derivatives in the reals hold in the complex realm.

Theorem 3.1. *If f and g are differentiable at z, then*

$$(f \pm g)'(z) = f'(z) \pm g'(z).$$

$$(cf)'(z) = cf'(z) \text{ (for any constant } c\text{).}$$

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z).$$

$$(f/g)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2} \text{ if } g(z) \neq 0.$$

If g is differentiable at z and f is differentiable at $g(z)$ then the chain rule holds

$$\frac{d}{dz} f(g(z)) = f'(g(z))g'(z).$$

We see that all of the same rules apply for derivatives in the complex and proofs can be found for these in [2]. Now, we will define an analytic function to be

Definition 3.2. *A complex-valued function $f(z)$ is said to be analytic on an open set G if it has a derivative at every point of G. Furthermore if $f(z)$ is analytic on the whole complex plane then it is said to be entire.*

3.2 Cauchy Riemann Equations

When taking the derivatives of complex functions, there seems to be a connection between the real and imaginary parts. This relationship can be easily derived and we replicate the derivation from [2] below. This will be a valuable asset in later sections. We start by letting Δz approach 0 from the right and top in equation (1) and we will end with the equations below.

Cauchy-Riemann Equations. *Let $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$, be differentiable, then*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proof: Let $f(z)$ be differentiable at $z_0 = x_0 + iy_0$ and define the function as $f(z) = u(x, y) + iv(x, y)$. Then the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) + f(z_0)}{\Delta z}$$

can be found by letting Δz go to 0 from any direction in the complex plane. For our purposes we will let it approach 0 horizontally, so $\Delta z = \Delta x$. Then we get

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_o) - u(x_0, y_0) + iv(x_0 + \Delta x, y_o) - iv(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{iv(x_0 + \Delta x, y_0) - iv(x_0, y_0)}{\Delta x}$$

These two limits are just the first partial derivatives of u and v with respect to x, so we can change this formula to

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \tag{2}$$

Now we want Δz to approach 0 vertically, so $\Delta z = i\Delta y$. We get the limit equation

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{iv(x_0, y_0 + \Delta y) - iv(x_0, y_0)}{i\Delta y}$$

and when we simplify, using the fact that $\frac{1}{i} = -i$, we get that

$$f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0). \tag{3}$$

If we set the real and imaginary parts of (2) and (3) equal to each other, we obtain our two equations. These will be very useful later in proving a function is analytic. QED

Theorem 3.2. *Let $f(z) = u(x, y) + iv(x, y)$ where $z = x + yi$, be defined in some open set G containing the point z_0. If the first partial derivatives of u and v exist in G, are continuous at z_0, and satisfy the Cauchy-Riemann equations at z_0, then f is differentiable at z_0.*

Proof: Take the difference quotient of f at z_0

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}, \tag{4}$$

where $z_0 = x_0 + iy_0$ and $\Delta z = \Delta x + i\Delta y$. The above expressions are well defined if $|\Delta z|$ is small enough that the closed disk with center z_0 and radius $|\Delta z|$ lies entirely in G. So (4) becomes

$$\frac{[u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)] + i[v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)]}{\Delta x + i\Delta y}. \tag{5}$$

We can rewrite the difference

$$u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)$$

as

$$[u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0+\Delta y)] + [u(x_0, y_0+\Delta y) - u(x_0, y_0)]. \tag{6}$$

Since the partial derivatives exist in G, by the mean value theorem, there is an x^* between x_0 and $x_0 + \Delta x$ such that

$$u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0+\Delta y) = \Delta x \frac{\partial u}{\partial x}(x^*, y_0+\Delta y).$$

Furthermore, since the partial derivatives are continuous at (x_0, y_0) we can write

$$\frac{\partial u}{\partial x}(x^*, y_0+\Delta y) = \frac{\partial u}{\partial x}(x_0, y_0) + \epsilon_1,$$

where $\epsilon_1 \to 0$ as $x^* \to x_0$ and $\Delta y \to 0$. Therefore equation (6) can be rewritten as

$$\Delta x[\frac{\partial u}{\partial x}(x_0, y_0) + \epsilon_1] + [u(x_0, y_0+\Delta y) - u(x_0, y_0)].$$

We can apply the same strategy to the right bracketed expression using ϵ_2 and get

$$u(x_0, y_0+\Delta y) - u(x_0, y_0) = \Delta y[\frac{\partial u}{\partial y}(x_0, y_0) + \epsilon_2].$$

Using this equation applying this same idea to the v-difference in equation (5), we get the equation

$$\frac{\Delta x[\frac{\partial u}{\partial x}(x_0, y_0) + \epsilon_1 + i\frac{\partial v}{\partial x}(x_0, y_0) + i\epsilon_3] + \Delta y[\frac{\partial u}{\partial y}(x_0, y_0) + \epsilon_2 + i\frac{\partial v}{\partial y}(x_0, y_0) + i\epsilon_4]}{\Delta x + i\Delta y}.$$

Now we use the Cauchy-Riemann equations to manipulate the partials on the right term in the numerator. It is important to note that $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$ and we can replace -1 with the term i^2 and therefore factor an i out of both terms.

$$\frac{\Delta x[\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)] + i\Delta y[\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)]}{\Delta x + i\Delta y} + \frac{\lambda}{\Delta x + i\Delta y}, \tag{7}$$

where $\lambda = \Delta x(\epsilon_1 + i\epsilon_3) + \Delta y(\epsilon_2 + i\epsilon_4)$. By the triangle inequality we have that

$$\left|\frac{\lambda}{\Delta x + i\Delta y}\right| \leq \left|\frac{\Delta x}{\Delta x + i\Delta y}\right| |\epsilon_1 + i\epsilon_3| + \left|\frac{\Delta y}{\Delta x + i\Delta y}\right| |\epsilon_2 + i\epsilon_4| \leq |\epsilon_1 + i\epsilon_3| + |\epsilon_2 + i\epsilon_4|.$$

Since $\epsilon_i \to 0$ for $i = 1, 2, 3, 4$, the last term in equation (7) goes to 0, and so

$$\lim_{x\to\infty} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

Therefore $f'(z_0)$ exists. QED

Lemma 3.3. *Any complex polynomial can be split into the sum of a real and imaginary polynomial.*

Proof: The general form of a complex polynomial is

$$P(z) = C_n(z)^n + C_{n-1}(z)^{n-1} + \cdots + C_1(z) + C_0$$

We know $z = x + iy$ so when replaced we get,

$$P(x, y) = C_n(x + iy)^n + C_{n-1}(x + iy)^{n-1} + \cdots + C_1(x + iy) + C_0,$$

then using binomial expansion, the equation will look like

$$P(x, y) = C_n(x^n + nx^{n-1}yi \cdots + nx(yi)^{n-1} + (yi)^n) +$$

$$C_{n-1}(x^{n-1} + (n-1)x^{n-2}yi \cdots + (n-1)x(yi)^{n-2} + (yi)^{n-1}) + \cdots + C_1(x + yi) + C_0.$$

Now it is easy to see that any term with an even power on the i term will be sent to the real part of the polynomial, since i to an even power is 1 or -1. Oppositely any term with a odd number on the i term will be sent to the imaginary part of the polynomial. If we factor an i out of every term on the imaginary side we end up with the equation

$$P(x, y) = R(x, y) + iQ(x, y)$$

where $R(x, y)$ is the real part of the polynomial and $Q(x, y)$ is the imaginary part. QED

Theorem 3.4. *Consider a polynomial $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ with $a_n \neq 0$. Then $f(z) = 1/P(z)$ is analytic everywhere except when $P(z) = 0$.*

Proof: $P(z)$ is a complex polynomial so using Lemma 3.3, we can split $P(z)$ into the sum of its real part $R(x,y)$ and its imaginary part $Q(x,y)i$, where $z = x+iy$. Therefore we end up with

$$\frac{1}{P(z)} = \frac{1}{R(x,y)+iQ(x,y)}, \tag{8}$$

where R and Q are real polynomials. We need to get the imaginary part out of the denominator so we manipulate equation (1) using the denominators conjugate.

$$\frac{1}{R(x,y)+iQ(x,y)}\frac{(R(x,y)-iQ(x,y))}{(R(x,y)-iQ(x,y))} = \frac{R(x,y)-iQ(x,y)}{R(x,y)^2+Q(x,y)^2}. \tag{9}$$

Now that we have the imaginary part out of the denominator, we can separate the function into its real and imaginary parts, $u(x,y)$ and $v(x,y)$ respectively.

$$u(x,y) = \frac{R(x,y)}{R(x,y)^2+Q(x,y)^2}, \quad v(x,y) = \frac{-Q(x,y)}{R(x,y)^2+Q(x,y)^2}. \tag{10}$$

The goal is to show that the Cauchy-Riemann equations are satisfied. First we'll see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. The two derivatives are:

$$\frac{\partial u}{\partial x} = \frac{(\frac{\partial R}{\partial x})(R(x,y)^2+Q(x,y)^2) - (R(x,y)(2R(x,y)) \times (\frac{\partial R}{\partial x}) + 2Q(x,y) \times (\frac{\partial Q}{\partial x}))}{(R(x,y)^2+Q(x,y)^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(-(\frac{\partial Q}{\partial y}))(R(x,y)^2+Q(x,y)^2) - (-Q(x,y)(2R(x,y)) \times (\frac{\partial R}{\partial y}) + 2Q(x,y) \times (\frac{\partial Q}{\partial y}))}{(R(x,y)^2+Q(x,y)^2)^2}$$

We see that the denominator is the same, so we set these equations equal to each other and simplify a little, and we get,

$$\left(\frac{\partial R}{\partial x}\right)(R(x,y)^2+Q(x,y)^2) - \left(R(x,y)\left(2R(x,y) \times \left(\frac{\partial R}{\partial x}\right) + 2Q(x,y) \times \left(\frac{\partial Q}{\partial x}\right)\right)\right)$$

$$= -\left(\frac{\partial Q}{\partial y}\right)(R(x,y)^2+Q(x,y)^2) + \left(Q(x,y)\left(2R(x,y) \times \left(\frac{\partial R}{\partial y}\right) + 2Q(x,y) \times \left(\frac{\partial Q}{\partial y}\right)\right)\right)$$

$$\iff$$

$$\begin{aligned} &-R(x,y)^2\left(\frac{\partial R}{\partial x}\right) + Q(x,y)^2\left(\frac{\partial R}{\partial x}\right) - 2R(x,y)Q(x,y) \times \left(\frac{\partial Q}{\partial x}\right) \\ &= Q(x,y)^2\left(\frac{\partial Q}{\partial y}\right) - R(x,y)^2\left(\frac{\partial Q}{\partial y}\right) + 2R(x,y)Q(x,y) \times \left(\frac{\partial R}{\partial y}\right). \end{aligned} \tag{11}$$

Also we know that since the function $P(z)$ is analytic, the Cauchy-Riemann equations imply that $\frac{\partial R}{\partial x} = \frac{\partial Q}{\partial y}$ and that $\frac{\partial Q}{\partial x} = -\frac{\partial R}{\partial y}$ therefore (11) changes to

$$-R(x,y)^2\left(\frac{\partial R}{\partial x}\right) + Q(x,y)^2\left(\frac{\partial R}{\partial x}\right) - 2R(x,y)Q(x,y) \times \left(\frac{\partial Q}{\partial x}\right)$$
$$= Q(x,y)^2\left(\frac{\partial R}{\partial x}\right) - R(x,y)^2\left(\frac{\partial R}{\partial x}\right) + 2R(x,y)Q(x,y) \times \left(-\frac{\partial Q}{\partial x}\right).$$

These are equal, so that means that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. Now we must check to see if $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

$$\frac{\partial v}{\partial x} = \frac{(-\frac{\partial Q}{\partial x})(R(x,y)^2 + Q(x,y)^2) - (-Q(x,y))(2R(x,y) \times (\frac{\partial R}{\partial x}) + 2Q(x,y) \times (\frac{\partial Q}{\partial x}))}{(R(x,y)^2 + Q(x,y)^2)^2}$$
$$-\frac{\partial u}{\partial y} = -\frac{(\frac{\partial R}{\partial y})(R(x,y)^2 + Q(x,y)^2) - (R(x,y))(2R(x,y) \times (\frac{\partial R}{\partial y}) + 2Q(x,y) \times (\frac{\partial Q}{\partial y}))}{(R(x,y)^2 + Q(x,y)^2)^2}.$$

We can simplify these two equations and set them equal to each other, also use the fact that $\frac{\partial Q}{\partial x} = \frac{\partial R}{\partial y}$ and $-\frac{\partial Q}{\partial y} = \frac{\partial R}{\partial x}$ to get that,

$$-R(x,y)^2\left(\frac{\partial R}{\partial x}\right) + Q(x,y)^2\left(\frac{\partial R}{\partial x}\right) - 2R(x,y)Q(x,y) \times \left(\frac{\partial Q}{\partial x}\right)$$
$$= Q(x,y)^2\left(\frac{\partial R}{\partial x}\right) - R(x,y)^2\left(\frac{\partial R}{\partial x}\right) + 2R(x,y)Q(x,y) \times \left(-\frac{\partial Q}{\partial x}\right).$$

Since these two equations are equal, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Therefore both Cauchy-Riemann equations are satisfied. The only time these equations fail is when

$$R(x,y)^2 + Q(x,y)^2 = 0.$$

This only happens when both polynomials $R(x,y)$ and $Q(x,y)$ are equal to 0, which means the polynomial $P(z)$ is equal to 0. Since we showed $1/P(z)$ is continuous everywhere except when $P(z) = 0$, then the function $1/P(z)$ is analytic everywhere except when $P(z) = 0$. QED

4 Complex Integration

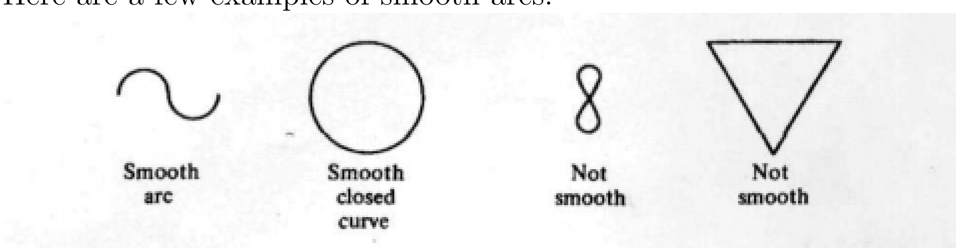
Section 4 will delve into the idea of complex integration progressing far enough to understand and prove Cauchy's Integral Formula. To be able to fully comprehend integrating with complex functions we must first understand different contours of these functions. This can be quite difficult to understand from definitions only, so in this first subsection numerous definitions and examples in the form of graphs have been taken from [2] to help illustrate what is being discussed.

4.1 Definitions

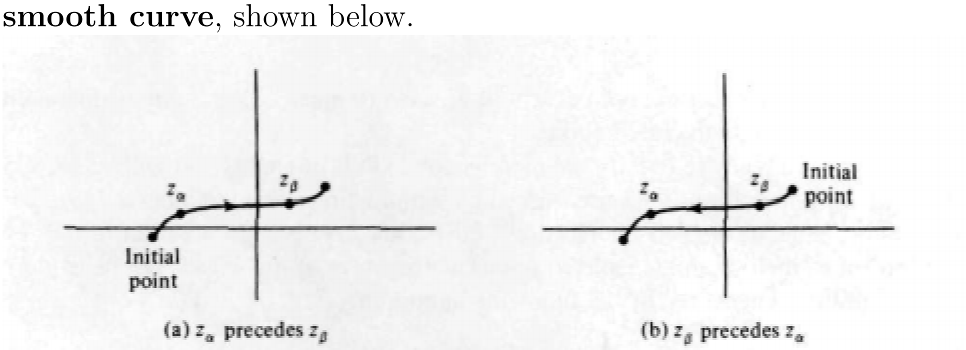
We will begin by defining what a smooth arc is.

Definition 4.1. *A point set γ in the complex plane is said to be a* ***smooth arc*** *if it is the range of some continuous complex-valued function $z = z(t),\ a \le t \le b$, that satisfies the following conditions:*
(i) $z(t)$ has a continuous derivative on $[a, b]$,
(ii) $z'(t) \neq 0$ for all $t \in [a, b]$,
(iii) $z(t)$ is one-to-one on $[a, b]$.
A point set γ is called a ***smooth closed curve*** *if it is the range of some continuous function $z = z(t), a \le t \le b$, satisfying conditions (i) and (ii) and the following (iii') $z(t)$ is 1-1 on the half open interval [a,b), but $z(b) = z(a)$ and $z'(a) = z'(b)$*

Here are a few examples of smooth arcs.



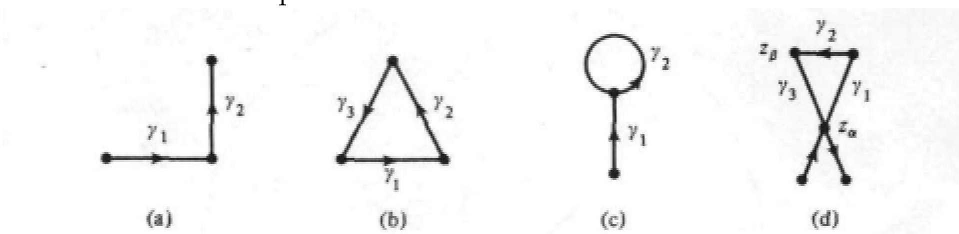
Next we add an initial point and direction to each curve to give us a **directed smooth curve**, shown below.

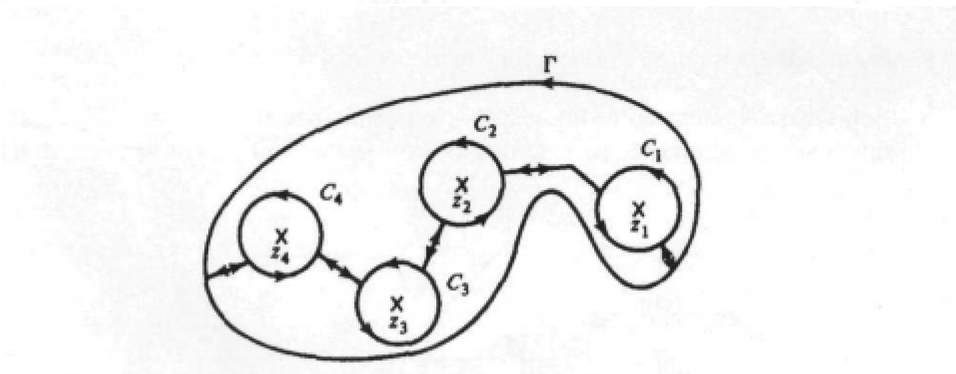


A **loop** occurs when the initial point and end point of a curve coincide, which will be used in later definitions.

Definition 4.2. *A* ***contour*** *Γ is either a single point z_0 or a finite sequence of directed smooth curves $(\gamma_1, \gamma_2, ...\gamma_n)$ such that the terminal point of γ_k coincides with* *the initial point of* γ_{k+1} *for each* $k = 1, 2, ..., n-1$. *We write this as* $\Gamma = \gamma_1 + \gamma_2 + ... + \gamma_n$.

Here are some examples of contours





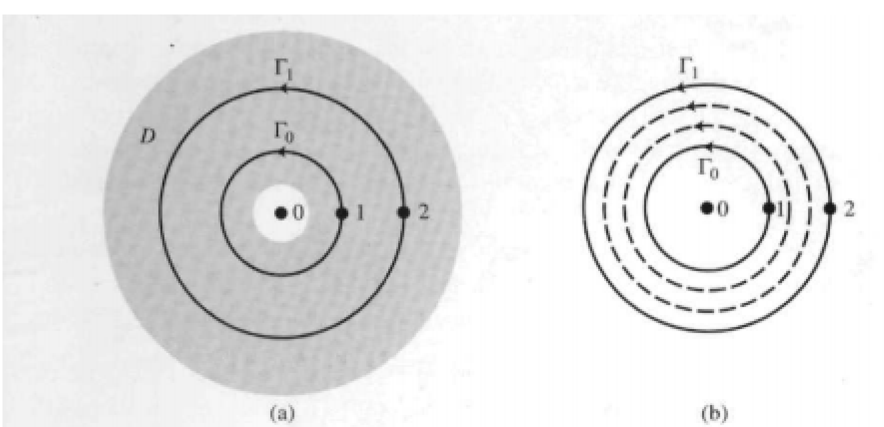
Definition 4.3. *Any domain* D *possessing the property that every loop in* D *can be continuously deformed in* D *to a point is called a* ***simply connected domain****.*

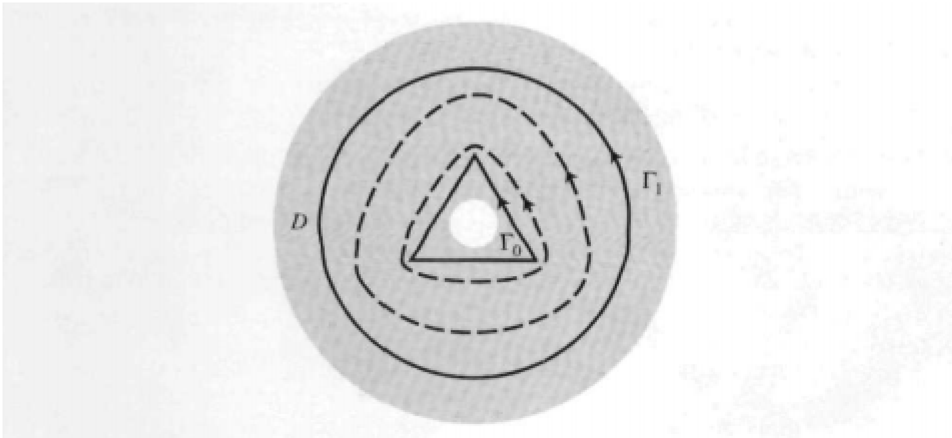
Definition 4.4. *The loop* Γ_0 *is said to be* ***continuously deformable*** *to the loop* Γ_1 *in the domain* D *if there exists a function* $z(s,t)$ *continuous on the unit square* $0 \leq s \leq 1, 0 \leq t \leq 1$, *that satisfies the following conditions:*

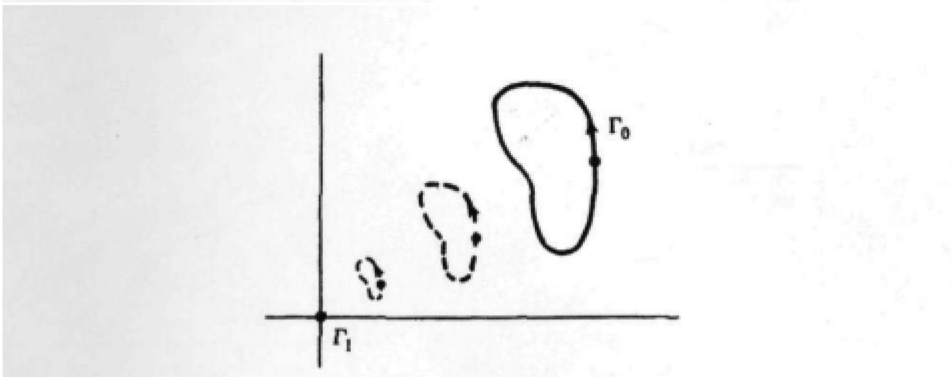
(i) For each fixed s *in* $[0,1]$, *the function* $z(s,t)$ *parametrizes a loop lying in* D.

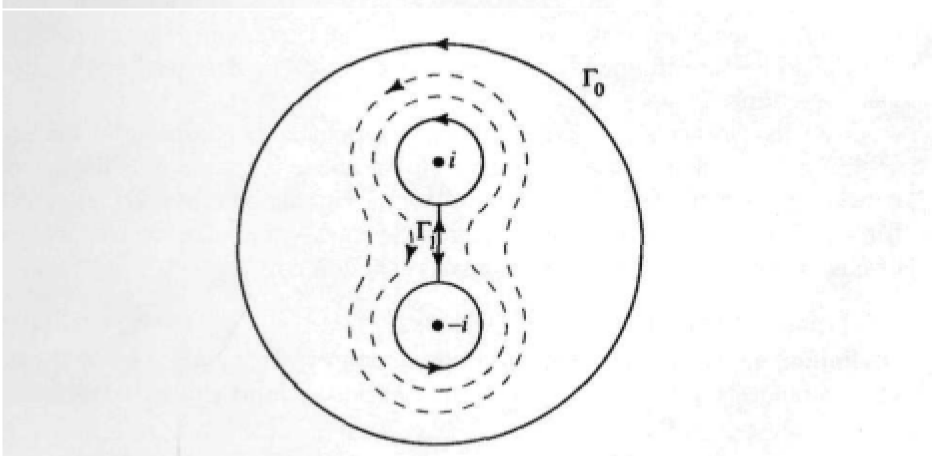
(ii) The function $z(0,t)$ *parametrizes* Γ_0

(iii) The function $z(1,t)$ *parametrizes the loop* Γ_1.









4.2 Integrals

Definition 4.5. *Suppose that* Γ *is a contour consisting of the directed smooth curves* $(\gamma_1, \gamma_2, ...\gamma_n)$*, and let* f *be a function continuous on* Γ*. Then the contour integral of* f *along* Γ *is denoted by the symbol* $\int_\Gamma f(z)dz$ *and is defined by the equation*

$$\int_\Gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + ... + \int_{\gamma_n} f(z)dz.$$

Now that we understand contours on the complex plane a little better, we can begin to understand how to integrate them.

Definition 4.6. *Let f be a complex-valued function defined on the directed smooth curve γ. Let z be the parameterization of γ from α to β where $z(\alpha) = a, z(\beta) = b$. Let P_n be a partition on γ where $\alpha = \alpha_0, \alpha_1, \alpha_2, ..., \alpha_n = \beta$. Let $\ell_v = z(\alpha_v^*)$ where $\alpha_v^* \in [\alpha_{v-1}, \alpha_v]$. For $S(P_n) = \sum_{v=1}^{n} f(\ell_v)(z_v - z_{v-1})$ where $z_v = \alpha_v$ we say that f is **integrable along** γ if there exists a complex number L that is the limit of every sequence of Riemann sums $S(P_1), S(P_2), \ldots, S(P_n), \ldots$ corresponding to any sequence of partitions of γ satisfying $\lim_{n\to\infty} \mu(P_n) = 0$; so*

$$\lim_{n\to\infty} S(P_n) = L \textit{ whenever } \lim_{n\to\infty} \mu(P_n) = 0$$

where $\mu(P_n)$ is the greatest difference between any consecutive points in the partition P_n

For the rest of the section we will use $\ell_v \in [z_{v-1}, z_v]$ in place of $\ell_v = z(\alpha_v^*)$ where $\alpha_v^* \in [\alpha_{v-1}, \alpha_v]$. This definition is quite similar to the definition of integration in real analysis because we paraemtrize the complex funxtion to a real function. Yet this does not provide us with enough information to be useful. We need the ability to integrate a gamma function between its boundary. It turns out the only conditions needed for this to be allowed are to have the contour γ be continuous and bounded. The proof of this requires concepts outside of complex analysis and a full description of the proof can be found in [1] on pages 109-110.

Theorem 4.1. *If f is continuous and bounded on the directed smooth curve γ for $a \le z \le b$, then f is integrable on γ and the integral is*

$$\int_\gamma f(z)dz = \int_a^b f(z)dz = \lim_{n\to\infty} \sum_{v=1}^{n} f(\ell_v)(z_v - z_{v-1}).$$

where

$$l_v \in [z_{v-1}, z_v]$$

.

Proof: Let D be a partition such that $a = z_0, z_1, \ldots z_n = b$. Then we can take a refinement on D' by taking a finite number of new points of division. There there exists a finite constant L such that $L_D = \sum_1^n |z_v - z_{v-1}| \leq L$. Then the sums satisfy the equation,

$$|\sum_D - \sum_{D'}| \leq O_D L$$

where O_D denotes the maximal oscillation of $f(z)$; i.e

$$O_D = \sup_{a,b \in D} |f(a) - f(b)|.$$

Now we will look at the first sub interval of the partition D, (z_0, z_1) and the term

$$f(\ell_1)(z_1 - z_0)$$

So we can say that this sub interval is divided by the partition D' into m sub intervals with $z_0 = z'_0, z'_1, \ldots z'_m = z_1$. Let a intermediate point of the interval (z'_{u-1}, z'_u) be denoted by $z = \ell'_u$. We can take the sum of these intervals and get

$$\sum_{u=1}^{m} f(\ell'_u)(z'_u - z'_{u-1}).$$

If we write

$$z_1 - z_0 = \sum_1^m (z'_u - z'_{u-1}),$$

then we have

$$|f(\ell_1)(z_1 - z_0) - \sum_1^m f(\ell'_u)(z'_u - z'_{u-1})|$$

$$= |\sum_1^m (f(\ell_1) - f(\ell'_u))(z'_u - z'_{u-1})| \leq O_D \sum_1^m |z'_u - z'_{u-1}|$$

We can apply this to every other subinterval of D and then n inequalities obtained this way can be added together. Using the triangle inequality we arrive at the inequality

$$|\sum_D - \sum_{D'}| \leq O_D L_D \leq O_D L.$$

Now lets take two arbitrary partitions D_1, D_2, with their respective sums being $\sum_{D_1}, \sum_{D_2}$. Therefore we can take a new partition D' which is a refinement of both D_1, and D_2. We can use this to find an inequality p–

$$|\sum_{D_1} - \sum_{D_2}| \leq |\sum_{D_1} - \sum_{D'}| + |\sum_{D_2} - \sum_{D'}| \leq L(O_{D_1} + O_{D_2}).$$

Since the function $f(z)$ is continuous on the closed arc ℓ, it is uniformly continuous on ℓ. Hence for any $\epsilon > 0$ there exists a number $\delta > 0$, such that the oscillation of O_D of the function $f(z(t))$ is less than ϵ for every partition D for which $t_v - t_{v-1} < \delta$. From the inequality $O_D < \epsilon$ it follows that for any two sufficiently fine partitions D_1, D_2

$$|\sum_{D_1} - \sum_{D_2}| \leq 2L.$$

Cauchy's convergence theorem now implies that the limit will always be finite and therefore the function $f(z)$ is integrable. QED

So now we know that if a function is continuous over an arc we can integrate it. The following proof can also be found in [1] on page 113, and is necessary for a future important proof.

Theorem 4.2. *Let f be a function continuous on the directed smooth curve γ. Then if $z = z(t), a \leq t \leq b$, is any admissible parametrization of γ consistent with its direction, we have*

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt. \tag{12}$$

Proof: For a partition D we can again take the sum

$$\sum_D = \sum_1^n f(z_{u-1})(z_u - z_{u-1}). \tag{13}$$

By the definition of the derivative,

$$z(t + \Delta t) - z(t) = z'(t)(\Delta t) + \Delta t(\Delta t)$$

and substituting Δt with $z_u - z_{u-1}$ this becomes

$$z_u - z_{u-1} = z'(t_{u-1})(t_u - t_{u-1}) + (t_u - t_{u-1})(\epsilon)$$

Now we want to substitute this equation into equation (13) to get

$$\sum_D = \sum_1^n f(z(t_{u-1}))(z'(t_{u-1})(t_u - t_{u-1}) + r_D) \tag{14}$$

where

$$r_D = \sum_1^n f(z(t_{u-1}))(z(t_u) - z(t_{u-1}))(\epsilon).$$

If we let $M = \max_{z\in\ell} w(z)$ then we can change the last equation into

$$|r_D| \leq M(b-a)\epsilon.$$

The function $f(t) = f(z(t))z'(t)$ is continuous in the interval (a,b). Therefore as the partition D becomes progressively finer, the first sum on the right side of equation (14) converges to the integral

$$\int_a^b f(t)dt = \int_a^b f(z(t))z'(t)dt.$$

Since the remainder term r_D converges to zero this integral is also the limit of our sum $\sum_D$ and therefore

$$\int_a^b f(z)dz = \int_a^b f(z(t))z'(t)dt,$$

QED

This next Theorem is very crucial to prove Cauchy's Integral Theorem. We need to show that if any of these contours are deformed into each other, then the integral of one contour equals the other. This is important to know when manipulating contours. A rigorous proof of this requires information from other advanced areas outside of complex analysis and can be found in [1]. Instead we will prove a weaker version that is useful for our purposes.

Theorem 4.3. *(Deformation Invariance Theorem) Let f be a function analytic in a domain D containing the loops Γ_0 and Γ_1. If these loops can be continuously deformed into one another in D, then*

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

Proof: We will prove a weaker version of this theorem. We will assume that there is a special case where Γ_0 and Γ_1 are linked by a deformation function $z(s,t)$ whose second order partial derivatives are continuous for $0 \leq s \leq 1, 0 \leq t \leq 1$ and $f'(x)$ is continuous. Now for each fixed s, the equation $z = z(s,t), 0 \leq t \leq 1$ defines a loop Γ_s in D. Let $F(s)$ be the integral of f along this loop, so that

$$F(s) = \int_{\Gamma_S} f(z)dz = \int_0^1 f(z(s,t))\frac{\partial z(s,t)}{\partial t}dt. \tag{15}$$

We want to take the derivative of $F(s)$ with respect to s. The assumptions above make it so that the integrand in (15) is continuously differentiable in s. We want to take the derivative of $F(s)$, so using the chain rule we get

$$\frac{dF(s)}{ds} = \int_0^1 \left[f'(z(s,t))\frac{\partial z}{\partial s} \times \frac{\partial z}{\partial t} + f(z(s,t))\frac{\partial^2 z}{\partial s \partial t}\right] dt. \tag{16}$$

We can also see that

$$\frac{\partial}{\partial t}\left[f(z(s,t))\frac{\partial z}{\partial s}\right] = f'(z(s,t))\frac{\partial z}{\partial t} \times \frac{\partial z}{\partial s} + f(z(s,t))\frac{\partial^2 z}{\partial s \partial t}.$$

This expression is the same as the integrand in (16), therefore

$$\begin{aligned}\frac{dF(s)}{ds} &= \int_0^1 \frac{\partial}{\partial t}\left[f(z(s,t))\frac{\partial z}{\partial s}\right] dt \\ &= f(z(s,1))\frac{\partial z}{\partial s}(s,1) - f(z(s,0))\frac{\partial z}{\partial s}(s,0).\end{aligned}$$

Since Γ_s is closed, we have $z(s,1) = z(s,0)$ for all s which have identical derivatives. Therefore $F(s)$ must be constant and $F(0) = F(1)$ and we can conclude that

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz.$$

QED

So it is possible now to integrate continuous loops. Here is an example that first deforms a contour into something that is usable and then integrates it using the methods developed earlier in the section.

Example Compute the integral $\int_{C_r} (z - z_0)^n dz$, with n an integer and C_r the circle $|z - z_0| = r$ traversed once in the counterclockwise direction.

***Solution**:* We want to parametrize C_r by using $z(t) = z_0 + re^{it}$, and $0 \le t \le 2\pi$. Let $f(z) = (z - z_0)^n$, then

$$f(z(t)) = (z_0 + re^{it} - z_0)^n = r^n e^{int},$$

and the derivative

$$z'(t) = ire^{it}.$$

Therefore by equation (12) and using substitution,

$$\int_{C_r} (z - z_0)^n = \int_0^{2\pi} (r^n e^{int})(ire^{it})dt = ir^{n+1}\int_0^{2\pi} (e^{it(n+1)})dt.$$

If $n \neq -1$ then

$$ir^{n+1}\int_0^{2\pi}(e^{it(n+1)})dt = ir^{n+1}\frac{e^{it(n+1)}}{i(n+1)}\Big|_0^{2\pi} = ir^{n+1}\left[\frac{1}{i(n+1)} - \frac{1}{i(n+1)}\right] = 0.$$

Now if $n = -1$ then

$$ir^{n+1}\int_0^{2\pi}(e^{it(n+1)})dt = i\int_0^{2\pi} dt = 2\pi i.$$

Therefore we know that

$$\int_{C_r}(z-z_0)^n dz = \begin{cases} 0 & \text{for } n \neq -1 \\ 2\pi i & \text{for } n = -1 \end{cases}$$

This example will be useful later on. Now we know enough information about complex integration to help us prove Cauchy's Integral Formula.

4.3 Cauchy's Integral Formula

Before solving Cauchy's Theorem we need to bring another concept over from real analysis into the complex plane. If a function has a positive upper bound M then the integral of the function will be less than M multiplied by the length of the arc. This is true in the complex plane too and can be proved fairly easily.

Theorem 4.4. *If f is continuous on the contour Γ and if $|f(z)| \leq M$ for all z on Γ, then*

$$\left|\int_\Gamma f(z)dz\right| \leq M\ell(\Gamma), \tag{17}$$

where $\ell(\Gamma)$ denotes the length of Γ.

Proof: Let f be a continuous function of a directed smooth curve γ and suppose that $f(z)$ is bounded by a constant M on γ for all z on γ. We want to take a Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta z_k$ with a partition $\mathcal{P}_n$ on γ. Then by the triangle inequality,

$$\left|\sum_{k=1}^{n} f(c_k)\Delta z_k\right| \leq \sum_{k=1}^{n}|f(c_k)||\Delta z_k| \leq M\sum_{k=1}^{n}|\Delta z_k|.$$

Also we know that the sum of lengths $\sum_{k=1}^{n} |\Delta z_k|$ cannot be greater than the entire length of γ, therefore

$$\left|\sum_{k=1}^{n} f(c_k)\Delta z_k\right| \leq M\ell(\gamma). \tag{18}$$

Inequality (18) is valid for all Riemann sums of $f(z)$, so by taking the limit as $\mu(\mathcal{P}_n) \to 0$ we get

$$\left|\int_{\gamma} f(z)dz\right| \leq M\ell(\gamma).$$

Now we apply the same process over the entire contour $\Gamma = \gamma_1 + \gamma_2 + ... + \gamma_n$. We start with the inequality from Definition 4.5, and we know that each Riemann sum on the contour γ_i is bounded by a constant M_i. We want to take M to be $M = \max\{M_1, M_2, ..., M_n\}$. Thus using the triangle inequality we get that

$$\left|\int_{\Gamma} f(z)dz\right| \leq M\ell(\Gamma).$$

QED

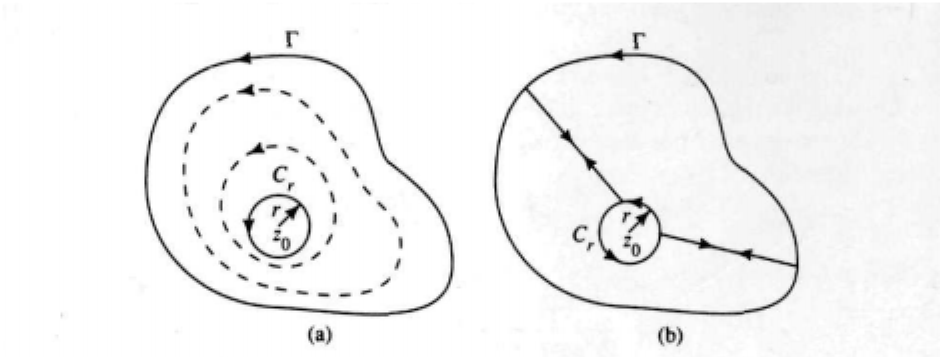
If a function is bounded, the integral of the function is also bounded. This can be a useful tool and we will put it to good use in the upcoming proof of Cauchy's Theorem.

Theorem 4.5. *(Cauchy's Integral Formula) Let Γ be a simple closed positively oriented contour. If f is analytic in some simply connected domain D containing Γ and z_0 is any point inside Γ, then*

$$f(z_0) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z - z_0}dz.$$

Proof: The function $\frac{f(z)}{z-z_0}$ is analytic everywhere in D except at z_0 since f is analytic everywhere. Therefore the contour Γ can be deformed into C_r, shown below, and furthermore an integral over Γ can be replaced with the integral over a small positively oriented circle $C_r : |z - z_0| = r$. So

$$\int_{\Gamma} \frac{f(z)}{z - z_0}dz = \int_{C_r} \frac{f(z)}{z - z_0}dz.$$



It helps here to express our new integral as the sum of two integrals:

$$\int_{C_r} \frac{f(z)}{z-z_0}dz = \int_{C_r} \frac{f(z_0)}{z-z_0}dz + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz.$$

We know from the example in section 4.2 that

$$\int_{C_r} \frac{f(z_0)}{z-z_0}dz = f(z_0)\int_{C_r} \frac{dz}{z-z_0} = f(z_0)2\pi i,$$

giving us

$$\int_{\Gamma} \frac{f(z)}{z-z_0}dz = f(z_0)2\pi i + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz. \tag{19}$$

Now from equation (19) we can see that only the last term is dependent on r so we allow the value r to go to 0.

$$\int_{\Gamma} \frac{f(z)}{z-z_0}dz = f(z_0)2\pi i + \lim_{r\to 0^+}\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz \tag{20}$$

Therefore if we can show the last limit is equal to 0 then the Cauchy equation follows. For this we want to take the maximum on C_r. Let $M_r = \max|f(z)-f(z_0)|$ for z on C_r. Then for z on C_r we get

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = \frac{|f(z)-f(z_0)|}{r} \le \frac{M_r}{r},$$

and by equation (17),

$$\left|\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz\right| \le \frac{M_r}{r}\ell(C_r) = \frac{M_r}{r}2\pi r = 2\pi M_r.$$

Since f is continuous at the point z_0, we know that $\lim_{r\to 0^+} M_r = 0$, and therefore

$$\lim_{r\to 0^+}\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}dz = 0.$$

So equation (20) reduces to

$$\int_{\Gamma} \frac{f(z)}{z-z_0}dz = f(z_0)2\pi i.$$

We divide each side by $2\pi i$ and we get Cauchy's Integral Formula. QED

5 Fundamental Theorem of Algebra

Now that we have Cauchy's Integral Theorem, we can prove a very integral Theorem to the subject of complex numbers. Now that we have define this we want to prove that all of these functions are entire.

Lemma 5.1. *All analytic functions are entire.*

Proof: This proof was first done by Cauchy using the function $\frac{1}{w-z}$. Let D be an open disk centered at a and suppose is differentiable everywhere within an open neighborhood of D. Let C be the positively oriented (counterclockwise) circle which is the boundary of D and let z be a point in D. Starting with Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw.$$

We then add and subtract a from the denominator to get

$$\frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)-(z-a)} dw.$$

Then factoring a $(w-a)$ out of the bottom it becomes

$$\frac{1}{2\pi i} \int_C \frac{1}{(w-a)} \times \frac{1}{1-\frac{z-a}{w-a}} f(w) dw.$$

It works out very nicely because the right fraction term is actually the sum of a geometric series so we can rewrite this equation as

$$\frac{1}{2\pi i} \int_C \frac{1}{(w-a)} \times \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a} \right)^n f(w) dw.$$

Interchanging of the integral and the infinite sum is allowed because $\frac{f(w)}{(w-a)}$ is bounded on C by some positive number M, so this becomes

$$\sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{(z-a)^n}{(w-a)^{n+1}} f(w) dw.$$

Note that $(z-a)^n$ does not depend on the variable of integration therefore we can pull it out and we end with the equation

$$f(z) = \sum_{n=0}^{\infty} (z-a)^n \frac{1}{2\pi i} \int_C \frac{1}{(w-a)^{n+1}} f(w) dw = \sum_{n=0}^{\infty} c_n (z-a)^n.$$

Thus you can write any analytic function as a power series, that is represented by the sum above, which looks like as polynomial. All polynomials are entire. QED

This proof is actually used in a very important theorem, namely that every entire function can be written as a power series.

Lemma 5.2. *Let f be analytic inside and on a circle C_R of radius R centered about z_0. If $|f(z)| \leq M$ for all z on C_R, then the derivatives of f at z_0 satisfy*

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n} \ (n = 1, 2, 3....).$$

Proof: Given that C_R has a positive orientation we have by Lemma 5.1 that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(w)}{(w - z_0)^{n+1}} dw.$$

For any w on C_R, the integrand is bounded by $\frac{M}{R^{n+1}}$ and the length of C_R is obviously $2\pi R$. By Theorem

refMaxbound it follows that

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R = \frac{n!M}{R^n}.$$

QED

Before we get to the Fundmental Theorem of Algebra. It must first be shown that any bounded entire function is constant. This is known as Liouville's theorem. Liouville's theorem was proved in 1847, and this proof is replicated from [2].

Theorem 5.3. *(Liouville's Theroem) The only bounded entire functions are the constant functions.*

Proof: Every entire analytic function $f(z)$ can be represented as a power series about 0:

$$f(z) = \sum_{k=0}^{\infty} c_k z^k,$$

where by Lemma 5.1

$$c_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_{C_r} \frac{f(z)}{z^{k+1}} dz,$$

where C_r is a circle about the origin with radius $r > 0$. If f is also bounded, then there exists a constant M such that $|f(z)| < M$ for all z. We can use this and Lemma 5.2 to see that

$$c_k = \frac{1}{2\pi i} \oint_{C_r} \frac{f(z)}{z^{k+1}} dz < \frac{1}{2\pi i} \oint_{C_r} \frac{M}{r^{k+1}} dz = \frac{M}{2\pi i r^{k+1}} \oint_{C_r} dz = \frac{M}{2\pi i r^{k+1}} 2\pi r = \frac{M}{i r^k}.$$

As $r \to \infty$, then for $k \geq 1$, all the terms c_k go to 0. Therefore the only term left is c_0, and $f(z) = c_0$, meaning that it is constant, proving all entire bounded functions are constant. QED

Now that we proved this we can finally prove The Fundamental Theorem of Algebra.

5.1 The Proof of the Fundamental Theorem of Algebra

Theorem 5.4. *Every non constant polynomial with complex coefficients has at least one zero.*

Proof: For the sake contradiction we take a general polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 \tag{21}$$

with $a_n \neq 0$, having no zeros. We know that $P(z)$ is entire. Since $P(z)$ has no zeros, by Theorem 3.4 $\frac{1}{P(z)}$ is entire. We want to show that $1/P(z)$ is bounded over the whole plane, so we manipulate (21) by factoring out a z^n. We get

$$P(z) = z^n (a_n + a_{n-1}/z + ... + a_1/z^{n-1} + a_0/z^n).$$

Then divide both sides by z^n and we get

$$P(z)/z^n = (a_n + a_{n-1}/z + ... + a_1/z^{n-1} + a_0/z^n). \tag{22}$$

Looking at the right side of the equation we can see that every term besides a_n has a z in the denominator so from (22) we can see that

$$P(z)/z^n \to a_n \text{ as } |z| \to \infty.$$

From that we know that $|P(z)/z^n| \geq |a_n|/2$ for $|z|$ sufficiently large or $|z| \geq M$ for some positive $M \in \mathbb{R}$. Therefore

$$|f(z)| = \left| \frac{1}{P(z)} \right| \leq \frac{2}{|z^n||a_n|} \leq \frac{2}{M^n |a_n|}.$$

We can see from this that $1/P(z)$ is bounded by $\frac{2}{M^n|a_n|}$ for $|z| \geq M$. If $|z| \leq M$ then $1/P(z)$ is bounded since z is in a compact set. Since $1/P(z)$ is bounded and entire it must be constant by Liouville's Theorem. If $1/P(z)$ is constant, then it follows that $P(z)$ is constant. If a function is constant then it can never cross over the zero axis unless $P(z) = 0$. Therefore, the only polynomials that have no zeros are constant functions. Therefore any non constant polynomials must have at least one zero.

QED

6 Applications

The most obvious idea to follow is that every polynomial of degree n has n roots.

Corollary 6.0.1. *Every polynomial with complex coefficients of degree n has n roots(counting multiplicity).*

Proof: Let $P_1(z) = C_n(z^n) + C_{n-1}(z^{n-1}) + \cdots + C_0$ be a polynomial with complex coefficients. By the Fundamental Theorem of Algebra $P(z)_1$ has a complex root. Therefore we can factor this root out evenly with no remainder and we get the equation

$$P_1(z) = (z - c_0)(K_{n-1}(z^{n-1}) + K_{n-2}(z^{n-2}) + \cdots + K_0).$$

The right multiple is a new polynomial $P_2(z)$ which can also have the Fundamental Theorem of Algebra applied to it. This process happens n times, and you end with the equation

$$(z - c_0)(z - c_1) \ldots (z - c_n).$$

Therefore the original polynomial $P_1(z)$ has n roots.

QED

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