Summer Research Projects for First Year Students

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Richter Scholar Program

- The Richter Scholar Summer Research Program provides students with the opportunity to conduct independent, individual research with Lake Forest College faculty early in their academic careers (have to be first year students).
- Students work one-on-one with a faculty member, doing independent research in one of a wide variety of fields. At the end of the program students present their work either via a poster or a presentation at the Richter Scholar Symposium.

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Projects

- What's so rational about the alphabet?
  Marina Rawlings and Kevin Kupiec.
  Summer 2014, three week project.
- Finding perfect polynomials mod 2.
  Ugur Caner Cengiz.
  Summer 2014, 10 week project.
- On Tupper's self-referential formula.
  Margaret Fortman.
  Summer 2015, 4 week project.

Projects Continued

- First-world solutions to First-year problems.
  Robert Mecham
  Summer 2015, 10 week project.
- Beatty sequences and the prime race.
  Noel Orwothwun
  Summer 2016, 4 week project.

Marina and Kevin

Background

- "Walking on Real Numbers" by Aragón, Bailey, Borwein and Borwein.
- Consider a number in base 4. For example π. In base 4,

$$\pi = 3.021003331222202020112203002031030 1\ldots,$$

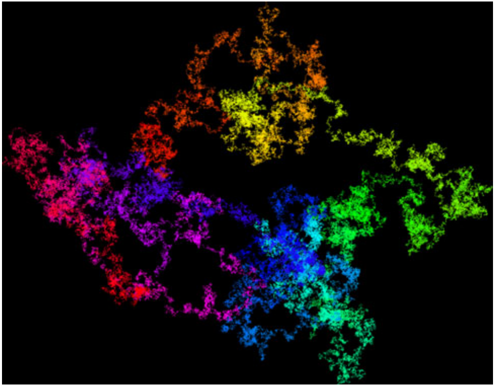
because

$$\pi = 3 + 0\left(\frac{1}{4}\right) + 2\left(\frac{1}{4^2}\right) + 1\left(\frac{1}{4^3}\right) + \ldots$$

- We start at the origin in the Cartesian plane. We move a unit to the right whenever we hit a digit 0, we move a unit up whenever we hit a digit 1, we move a unit left whenever we hit a digit 2 and we move a unit down if we hit a digit 3.

Walking on π

Walking on the first 100 billion digits of π reveals the following picture:



Easier Example

Walking on the number 419636198 which can be rewritten in base 4 as

$$121000302033212_4.$$

Inspiration

1049012271677499437486619280565448601617567358491560876166848380843144358447
2528755516292470277595555704537156793130587832477297720217708181879659063736
5767487981422801328592027861019258140957135748704712290267465151312805954195
39975042020613803738223389597133919 54

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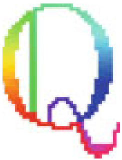
1612226962694290912940490066273549214229880755725468512353395718465191353017
3488143140175045399694454793530120643833272670970079330526292030350920973600
4509554561365966493250783914647728401623856513742952945308961226815274887561
565807616241078807518459942193877 4835



Project

- For each letter of the alphabet, find a rational that satisfies that if you random walk through that rational, you get the letter of the alphabet.
- Build a computer program that can use the information above to figure out the rational that works for a particular phrase.

Alphabet

How do you find the rational for a particular letter?

- Find a string of digits that "spell out" your letter in such a way that you end up where you started. For example, for the letter "D", it would be 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 0, 3, 0, 1, 2, 0, 3, 0, 3, 2, 0, 0, 3, 2, 0, 0, 3, 2, 0, 3, 2, 0, 0, 3, 2, 0, 3, 2, 3, 0, 2, 3, 0, 3, 2, 0, 0, 3, 2, 2, 3, 0, 0, 3, 2, 2, 1, 3, 0, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 3, 2, 1, 3, 3, 0, 1, 3, 2, 2, 3, 0, 0, 1, 3, 2, 2, 3, 0, 0, 3, 2, 2, 0, 3, 2, 3, 0, 2, 3, 0, 2, 3, 2, 0, 3, 2, 2, 0, 3, 2, 0, 3, 2, 2, 0, 3, 2, 2, 0, 3, 2, 1, 2, 0, 3, 2.
- Consider the dot product with $\{1/4, 1/4^2, 1/4^3, \ldots, \ldots\}$. In our example we'd have

$$1\left(\frac{1}{4}\right) + 0\left(\frac{1}{4^2}\right) + \ldots =$$

$$\frac{6384779382043951036217348661253680515005885357484535471589654514956414794662721006368542597248986985323127416704519810815261318970154183}{2325883917745942049757836185241614509931652354199417792900768637378045721962873354643811362284043409794440069140051769387310725211566899 2}$$

Alphabet Continued

One issue with just finding the rational as above, is that the random walk will now continue indefinitely to the right as the expansion ends with infinitely many zeroes. To fix this, we consider the number of digits in the representation of the letter and then do a geometric series expansion.
For example with the letter "D", it has 227 digits. Let the rational representation be x. Then the rational that loops itself over and over would be

$$x + 4^{-227}x + \left(4^{-227}\right)^2 x + \ldots = \frac{1}{1 - 4^{-227}}x.$$

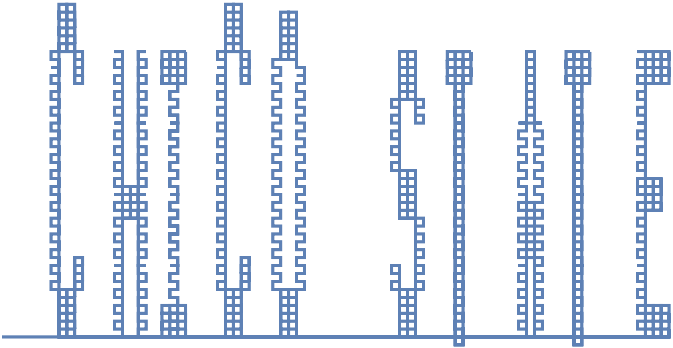
Example: Numerator has 1461 digits

641854286382104122265200073944393190340011436268004432577571447976969296450247665751776272
405057553827438925969290051756674576913488858091263786508005725144276413385765710561587363
482139307381096359854742736246056364606411853710240210838309635777126658624141964405801697
538761519303363624429506433569139845141303413401779804700444621146008340211436562946673232
391449313491972565702387067992396816596732432949838189092600300584534318766223291315042934
725070960316514014092307814602625365423361837293666346883059989463098708376701624704016978
803920535031065200711464682288735228769534369356957599536887159812053111225385766451323878
188436140934811052537676417819418096135418723005929009328849285700633999117150205240088751
436606031527467025385772739648490580316687900199530319421020007627044859727775885970673654
507849324360814426805941741060561607691708491819055405764931955346124283322581158236157221
647832036887579323344110548422155275728546356602230968111016031896238708987904261441315730
292424096621206710097312614206010151004755516130368571079892634697115770590012588338346791
907141245104520040867315687457589934523478684386659059913639838770608247772651904954071923
587866884617092293524233431341983072323511438055096817982472926085203992537079637812547170
498573974635154906106088204024068317390622842775882155790558573409725108109463836973194740
508644793111989536495578610726887955329063004411414975846404525187256322216114075385473674
397739904630184354338

/

126674424514970223578537504466118411976171966868895503730665232632837902735682541653417876
014878156372508169110744841821162915521622670631754088356466641623019019896389625333691652
770813774391031103664248807344356746245382899693357312535457656424716306662682308343301591
610242630147003082003497030984505342763834897960485862185726309723618848617291422317922519
689528909828938766133075583223177391564592981775308733440193688654151400521011091540140068
004099158830291243223810828297429069672268063243642269348274853173524309207744752729606685
517343686551106782577609822587577670721338720441411298211430894226110142126271669400020785
789054681782589918224694756060272014028507690568222689819111417380817736549063340661007182
301403471614218259048720878397961323898405004580471459935392301986846708399220821016928625
179476168461504765985153745313990183725439856182952272182114412285739778784640885518530351
298132877724875117979412822233762171861896273928052890479778305938020881793196095157805917
743773674832775857746400386259283092430319132653709685036171551286454535369295323384086146
386433517745574077095639731772763858146540261995404484359857682806220026524930209367091528
052689296670137048399770680431732046653458493733812635526497390599745948173668077993555764
934524227737565940726399795462669585610790128091484728575376256223772829027140290308793426
973812948111151005630998698365112200166278554373730556311050253932991586142265680272136632
229772654437250860318 7

Picture

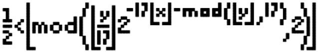


Margaret

Tupper's self-referential formula

$$\frac{1}{2} < \left\lfloor \operatorname{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \operatorname{mod}(\lfloor y \rfloor, 17)}, 2 \right) \right\rfloor$$

Graph of the above equation for $0 \leq x \leq 105$ and $k \leq y \leq k + 17$ is



for k equal to
4858450636189713423582095962494202044581400587983244549483093085061934704708809
92845064476986552436484999724702491511911041160573917740785691975432657185544205721
04457358836818298237541396343382251994521916512843483329051311931999535024137587652
39264874613394906870130562295813219481113685339535565290850023875092856892694555974
281546386510730049106723058933586052544096664351265349363643957125556569593681518433
48576052669401612512669514215505395545191537854575257565907405401579290017659679654
80064427829131488548259914721248506352686630476300

Project

For a given sentence, find the integer k such that the graph of Tupper's formula looks like that sentence for $0 \leq x \leq 105$ and $k \leq y \leq k + 17$.

Example

Plot of Tupper's formula for $0 \leq x \leq 105$ and $k \leq y \leq k + 17$ when k is

24664157472184423239649945387279224708508126841056624618784335869967101252215626708735620078567681519603670424745830399840776135440157097120379786269794329012635847408189607003421108454152989619210652798167085613652408864020789698828961541994770782453977248093324486094638246116719044042543908310742866615477934678465501917671424

How to get it done

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	1	1	1	1	1
0	0	0	0	0	0	0	0	1	0	0	1	0	1	0
0	0	0	0	0	0	0	0	1	0	0	1	0	1	0
0	0	0	0	0	0	0	0	1	1	0	1	0	1	0
0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

As the previous project, the key is figuring out how to do a letter first.

1	1	1
0	0	1
1	1	1
1	0	1
1	1	1

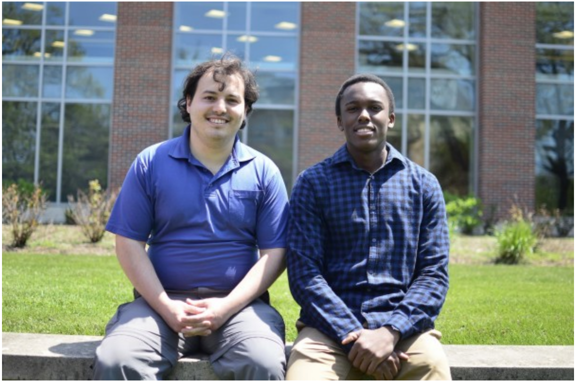
Letter a

1	1	1
0	0	1
1	1	1
1	0	1
1	1	1

The binary number for the letter a is 11101 10101 11111
Multiply by 2^{17} to move column to the right.
Formula for the lowercased a is:

$$17((1+2+4+16)+(1+4+16)2^{17}+(1+2+4+8+16)2^{34})$$

Noel



Beatty Sequences

- Given a positive irrational number θ, the Beatty sequence associated to θ is the sequence of integers:

$$\lfloor \theta \rfloor, \lfloor 2\theta \rfloor, \ldots$$

- If two positive irrational numbers α, β satisfy $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then we say that the Beatty sequences for α and β are complementary since their union is $\mathbb{N}$ and the intersection is empty.
- For the rest of this section, let A be the Beatty sequence associated to α and B the one associated to β. Then

$$A \cup B = \mathbb{N} \quad A \cap B = \emptyset.$$

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Primes in Beatty Sequences

- A theorem of Ribenboim says that the density of primes that land on A is $\frac{1}{\alpha}$. So we expect that among the first n primes, $\frac{n}{\alpha}$ of them land in A and $\frac{n}{\beta}$ of them land in B.
- The question is whether there's any bias in one direction.
- For the rest of the talk, we'll just consider a value for α and deduce β in terms of α (since $\frac{1}{\alpha} + \frac{1}{\beta} = 1$).

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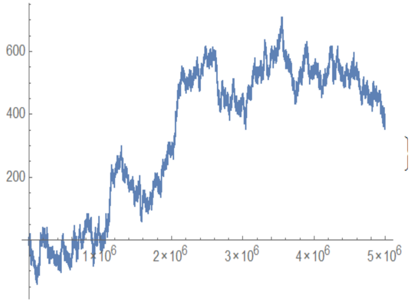

Figure: Difference for $\alpha = \sqrt{2}$ up to 5000000

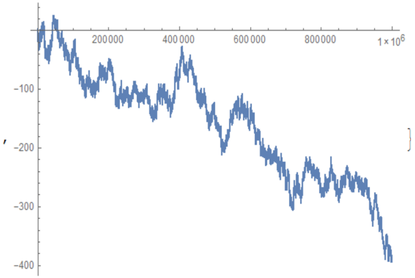

Figure: Difference for $\alpha = \sqrt{3}$ up to 1000000

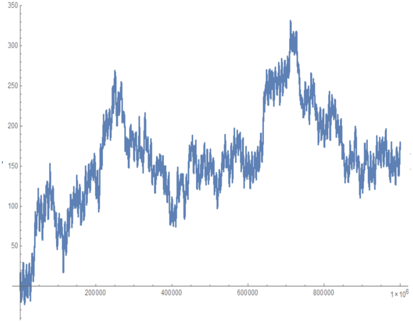

Figure: Difference for $\alpha = \pi$ up to 1000000

Bias between consecutive primes

- Soundararajan and Lemke-Oliver recently (this year!) discovered that there's a repulsion among the last digit of consecutive primes. For example if a prime ends in the digit 1, the next prime is less likely to end with digit 1 than randomness suggests.
- This phenomenon happens in any modulus. We'll focus on mod 3 where primes (other than 3) have two possibilities, i.e., 1 mod 3 and 2 mod 3.
- Is there such a bias with Beatty sequences?

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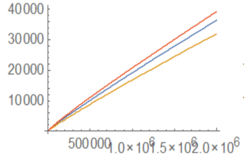
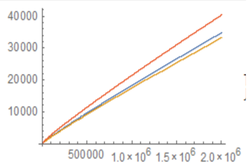
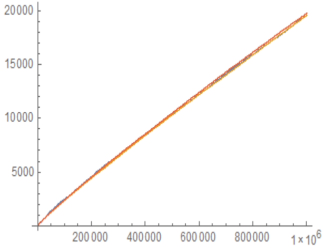
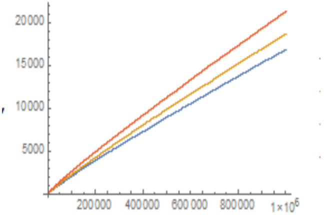
In the following pictures, the red color represents when the consecutive primes land on different Beatty sequences, the blue color represents when they both land on A, and the yellow color when they both land on B.
The first graph is for $\alpha = \sqrt{2}$, the second is for $\alpha = \sqrt{3}$ for numbers up to 2000000.





These ones are for $\alpha = \pi$, and $\alpha = e$ for numbers up to 1000000.





Chebyshev Bias

- Chebyshev noticed that primes 1 modulo 3 seem to appear less often than primes 2 modulo 3.
- It turns out that in general, if a is such that there is an x with $x^2 \equiv a \bmod n$, then primes modulo a appear less frequently.
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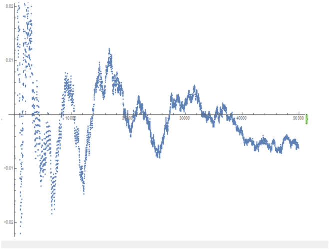

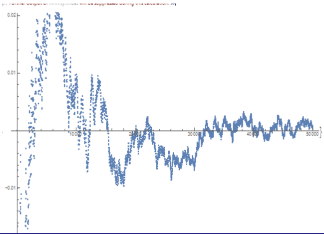

Robert

First year studies

- At Lake Forest College every incoming first year student gets assigned a "first-year-studies" course.
- Students rank their choices.
- The administration then decides where to send students depending on several factors:
  1. All classes should have roughly the same number of students.
  2. Some classes have an ACT/SAT threshold.
  3. Some classes do not allow athletes (because the classes do many field trips).
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Answer

Yes.

Process

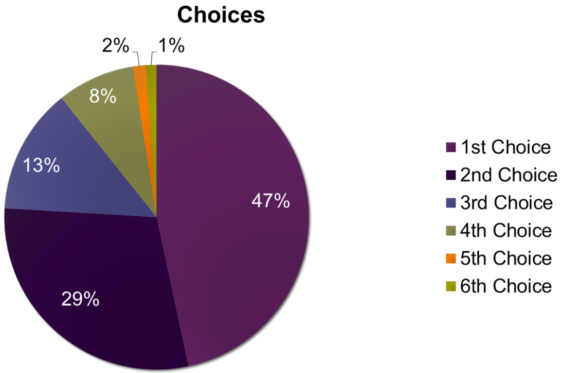
- Start out small, work with toy problems and familiarize yourself with the language and problem
- Gather the data needed and break it down
- Separate the data into two main groups groups: Students, Classes
- Divide the two main groups into even smaller groups: males/females, athletes, ACT scores, Non-athlete classes, Classes with certain ACT scores
- Make minor changes to your toy problem and feed the data into the main code
- Analyze the outcome and rejoice in a job well done!

Mishaps

There were several things that held us back for days and sometimes weeks

- Missing and extra numbers in the files
- Classes that had few, if any, people pick them
- Student-athletes picking non-athlete classes
- Misspelling of classes
- BLANK SPACES!!!!!!!!!

Results


Choices
2%
1%
8%
13%
47%
29%
1st Choice
2nd Choice
3rd Choice
4th Choice
5th Choice
6th Choice

Things Robert learned (according to him)

- A new programming language that will be of use as I continue in the math/CS fields
- You can never check your data too many times
- Always start out small and double check each step
- Solving a problem no matter the size one of the greatest feelings

Caner

Perfect Numbers

$$6 = 1 + 2 + 3$$

$$28 = 1 + 2 + 4 + 7 + 14.$$

Polynomials Mod 2

- A polynomial mod 2 is one of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

  where $a_i \in \{0, 1\}$.
- We consider the operation mod2, i.e., $1 + 1 = 0, 0 + 1 = 1 + 0 = 1, 0 + 0 = 0$.
- For example

$$x^2 + 1 = x^2 + 2x + 1 = (x + 1)^2.$$

Perfect Polynomials Mod 2

- Let $\sigma(P)$ be the sum of the divisors of a polynomial P in mod 2.
- A polynomial is said to be perfect mod 2 if $\sigma(P) = P$.
- $x^2 + x = x(x+1)$, so
$$\sigma(x^2 + x) = 1 + x + (x+1) + x^2 + x = x^2 + x.$$
So it is perfect.
- $$\sigma(x^2 + 1) = 1 + (1 + x) + (1 + x^2) = 1 + x + x^2.$$

Degree	Factorization into Irreducibles
5	$T(T+1)^2(T^2+T+1)$
	$T^2(T+1)(T^2+T+1)$
11	$T(T+1)^2(T^2+T+1)^2(T^4+T+1)$
	$T^2(T+1)(T^2+T+1)^2(T^4+T+1)$
	$T^3(T+1)^4(T^4+T^3+1)$
	$T^4(T+1)^3(T^4+T^3+T^2+T+1)$
15	$T^3(T+1)^6(T^3+T+1)(T^3+T^2+1)$
	$T^6(T+1)^3(T^3+T+1)(T^3+T^2+1)$
16	$T^4(T+1)^4(T^4+T^3+1)(T^4+T^3+T^2+T+1)$
20	$T^4(T+1)^6(T^3+T+1)(T^3+T^2+1)(T^4+T^3+T^2+T+1)$
	$T^6(T+1)^4(T^3+T+1)(T^3+T^2+1)(T^4+T^3+1)$

Thank you!