The Mathematics of Consociational Democracy

Arthur J. Brown and Finnian J. Bunta and Enrique Treviño

Department of Mathematics, Lake Forest College

Motivation

Consociational Democracy

- Arend Lipjhart, 1970s
- Deeply divided blocs along social, ethnic, or religious lines
- Blocs divided into citizenry (C) and elites, or polity (P)
- Elites communicate between blocs, citizens do not
- Netherlands, Belgium, Lebanon, and recently Bosnia

The Model

Boynton and Kwon formed a model based on five assumptions:

1. The political actors are:
   (a) divided in blocs
   (b) within blocs they are divided into elites and citizens
2. There is no dialogue between the citizenry of different blocs.
3. The elite engage in political decision making by accommodation, forming a "grand coalition."
4. The elites of each bloc are independent of the citizens of each bloc.
5. The different blocs respond to the same political issue in different ways, forming different reactionary opinions.

This gives, in the two-party case, the following system:

$$\begin{aligned} C_1'(t) &= \alpha_{10}[P_1(t) - C_1(t)] + \beta_{01}U(t) \\ C_2'(t) &= \alpha_{20}[P_2(t) - C_2(t)] + \beta_{02}U(t) \\ P_1'(t) &= \alpha_{01}[C_1(t) - P_1(t)] + \alpha_{21}[P_2(t) - P_1(t)] + \beta_1 U(t) \\ P_2'(t) &= \alpha_{02}[C_2(t) - P_2(t)] + \alpha_{12}[P_1(t) - P_2(t)] + \beta_2 U(t) \end{aligned} \tag{1}$$

And the inequalities:

$$\alpha_{01} < \alpha_{10}, \qquad \alpha_{02} < \alpha_{20}, \qquad \alpha_{01}, \alpha_{02} < \alpha_{12}, \alpha_{21}$$

.

Generalizing

' In order to make the coefficients more intuitive and useful for generalized versions of the problem, they were redefined as follows:

$$\alpha_{ij} \text{ is the coefficient on } \begin{cases} [P_i(t) - P_j(t)] & \text{if } i \neq j, i, j \neq 0 \\ [C_i(t) - P_i(t)] & \text{if } i = 0 \\ [P_j(t) - C_j(t)] & \text{if } j = 0 \end{cases}.$$

β_{0i} is the coefficient on $U(t)$ for $C_i'(t)$.

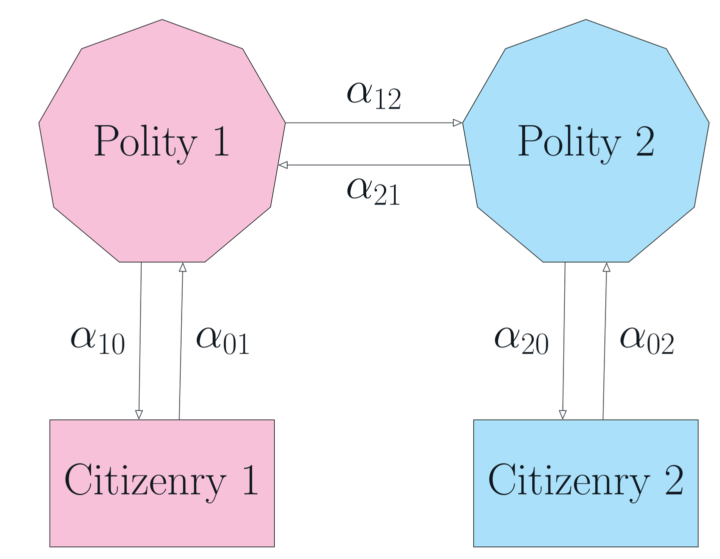
β_i is the coefficient on $U(t)$ for $P_i'(t)$.

Generally, the equations are as follows:

$$C_i'(t) = \alpha_{i0}[P_i(t) - C_i(t)] + \beta_{0i}U(t)$$
$$P_i'(t) = \alpha_{0i}[C_i(t) - P_i(t)] + \alpha_{1i}[P_1(t) - P_i(t)] + \alpha_{2i}[P_2(t) - P_i(t)] + \ldots + \beta_i U(t)$$
$$\alpha_{0i} < \alpha_{i0}, \qquad \alpha_{0i} < \alpha_{ji}$$

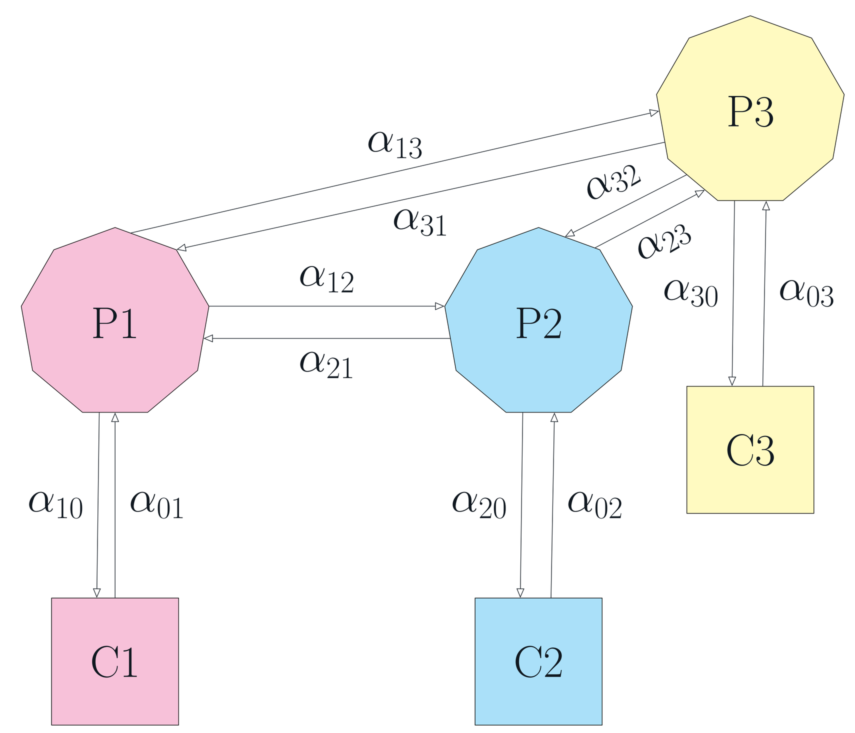
.

2-Party Model



3-Party Model

$$\begin{aligned} C_1'(t) &= \alpha_{10}[P_1(t) - C_1(t)] \\ C_2'(t) &= \alpha_{20}[P_2(t) - C_2(t)] \\ C_3'(t) &= \alpha_{20}[P_2(t) - C_2(t)] \\ P_1'(t) &= \alpha_{01}[C_1(t) - P_1(t)] + \alpha_{21}[P_2(t) - P_1(t)] + \alpha_{31}[P_3(t) - P_1(t)] \\ P_2'(t) &= \alpha_{02}[C_2(t) - P_2(t)] + \alpha_{12}[P_1(t) - P_2(t)] + \alpha_{32}[P_3(t) - P_2(t)] \\ P_3'(t) &= \alpha_{03}[C_3(t) - P_3(t)] + \alpha_{13}[P_1(t) - P_3(t)] + \alpha_{23}[P_2(t) - P_3(t)] \end{aligned} \tag{2}$$



Matrix-Vector Equation

This system of equations can be represented by a matrix-vector equation in the form $\vec{x}' = A\vec{x}$, by making the following substitutions:

$$x_{01} = C_1(t) - P_1(t),$$
$$x_{02} = C_2(t) - P_1(t),$$
$$x_{12} = P_1(t) - P_2(t),$$

This gives us

$$\frac{d}{dt}\begin{bmatrix} x_{01} \\ x_{02} \\ x_{12} \end{bmatrix} = \begin{bmatrix} -(\alpha_{01} + \alpha_{10}) & 0 & \alpha_{21} \\ 0 & -(\alpha_{02} + \alpha_{20}) & -\alpha_{12} \\ -\alpha_{01} & -\alpha_{02} & -(\alpha_{12} + \alpha_{21}) \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ x_{12} \end{bmatrix}$$

The object of our research was to show that the eigenvalues of this matrix had negative real part, implying stability under certain circumstances for the differential equations.

Viete's

Another intriguing result reached was that, at least in the 2-bloc case, that the eigenvalues are real implies that they are negative. This can be proved using Viete's formulas for the cubic polynomial $ax^3 + bx^2 + cx + d$:

$$r_1 + r_2 + r_3 = -\frac{b}{a}, \quad r_1 r_2 + r_1 r_2 + r_2 r_3 = \frac{c}{a}, \quad r_1 r_2 r_3 = -\frac{d}{a}$$

In our case, $-\frac{b}{a}$ is negative, $\frac{c}{a}$ is positive, and $-\frac{d}{a}$ is negative. Try to see how this implies each of r_1, r_2, and r_3 is negative.

Results

A number of interesting results cropped up throughout the course of our research, some of them are listed below.

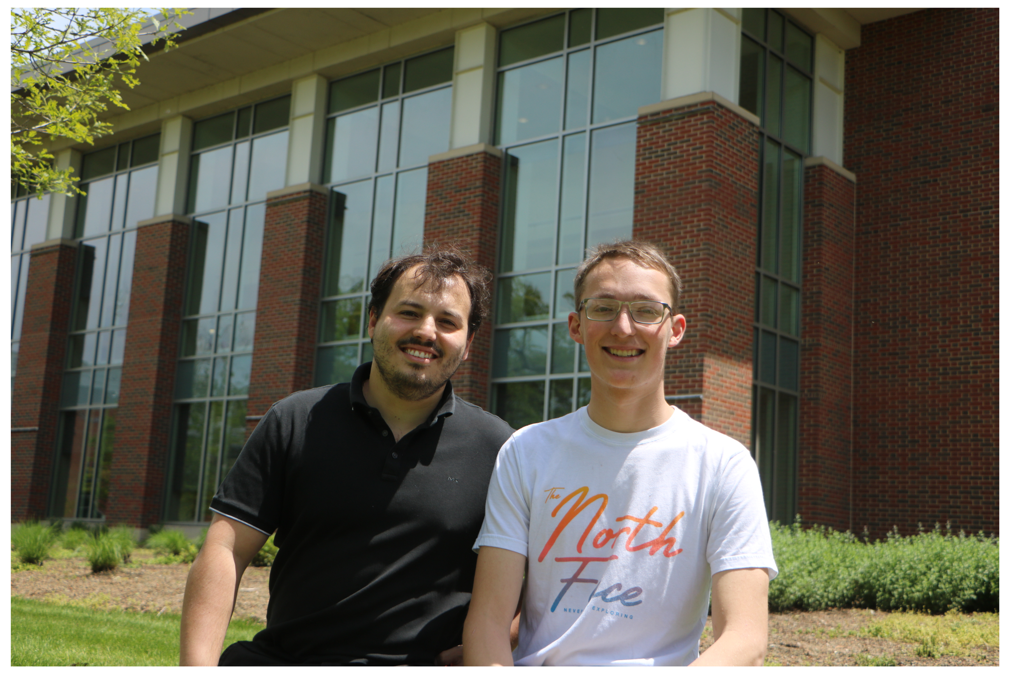
- If $\lim_{t\to\infty} U(t) = 0$, the system is stable.
- If $U(t)$ is bounded, the groups' views have bounded differences.
- We cannot prove the first result for 3-bloc matrices.

Acknowledgements

I would like to extend my gratitude first and foremost to my mentor Enrique Treviño for presenting the problem, offering invaluable help in the problem-solving process, and generously donating his time to the Richter Scholar program.

I would like to thank my roommates for many a fun night of Coup, Catan, and Codenames.

And I'd like to extend my utmost appreciation to Shubhik DebBurman, Zubair Mohammed, Lexi Dejneka, and Lake Forest College for this opportunity.



References

[1] G. R. Boynton and W. H. Kwon. "An Analysis of Consociational Democracy". In: *Legislative Studies Quarterly* 3 (Feb. 1978), pp. 11–25.

[2] Matthew Chakov. *A Look at Consociationalism in Bosnia and Herzegovina*. `https://edspace.american.edu/theworldmind/2017/04/24/an-exotic-yet-surprisingly-appropriate-government-a-look-at-consociationalism-in-bosnia-and-herzegovina/`. Accessed: 2019-06-3.

[3] Reut Institute. *Consociationalism*. `http://reut-institute.org/en/Publication.aspx?PublicationId=3207`. Accessed: 2019-06-5.