An unusual recursive formula to answer a question regarding fixed points in permutations

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Suppose n people leave a coat at the coat check at a party. Upon exiting, the coats are distributed back randomly. What is the average number of people who get their coat back? This classical question is usually phrased as "what is the expected value of fixed points in a permutation on n objects?". It can be easily answered using linearity of expectation [**3**, p. 265]. Here we give a non-probabilistic proof that relies on some nice recursive formulas for the number of derangements d_n [**1**]. The proof will use the following recursive formula for d_n that is not well-known:

$$d_n = \sum_{k=1}^{n} (k-1)\binom{n}{k} d_{n-k}. \tag{1}$$

Equation (1) seems to have been first proved by Deutsch and Elizalde [**2**]. Deutsch and Elizalde provide two proofs of (1), a bijective proof and a proof using generating functions. In the next section we give our own proof using induction on Euler's recurrence relation $d_n = (n-1)(d_{n-1} + d_{n-2})$ [**3**, p. 85-86].

An unusual recursive formula

Our goal is to prove (1). We start with Euler's recurrence relation for derangements:

$$d_n = (n-1)(d_{n-1} + d_{n-2}). \tag{2}$$

If we define $d_{-1} = 0$, then (2) is true for any $n \geq 1$. From this recursion, we can prove by induction that for all $2 \leq m \leq n$,

$$d_n = \sum_{k=2}^{m} (k-1)\binom{n}{k} d_{n-k} + \binom{n-1}{m} d_{n-m} + (n-m)\binom{n-1}{m-1} d_{n-m-1}. \tag{3}$$

Indeed if we assume the above to be true for a particular $m \leq n-1$, then by replacing d_{n-m} using (2) in the term $\binom{n-1}{m} d_{n-m}$ we get

$$\begin{aligned} d_n = &\left(\sum_{k=2}^{m} (k-1)\binom{n}{k} d_{n-k}\right) + (n-m-1)\binom{n-1}{m} d_{n-m-1} \\ &+ (n-m)\binom{n-1}{m-1} d_{n-m-1} + (n-m-1)\binom{n-1}{m} d_{n-m-2}. \end{aligned} \tag{4}$$

Now, using that $\binom{n-1}{m} + \binom{n-1}{m-1} = \binom{n}{m}$, we have

$$(n-m-1)\binom{n-1}{m} + (n-m)\binom{n-1}{m-1} = (n-m)\binom{n}{m} - \binom{n-1}{m}. \tag{5}$$

By using $(n-m)\binom{n}{m} = (m+1)\binom{n}{m+1}$ in (5), we get

$$\begin{aligned}(n-m)\binom{n}{m} - \binom{n-1}{m} &= (m+1)\binom{n}{m+1} - \binom{n-1}{m} \\ &= m\binom{n}{m+1} + \binom{n}{m+1} - \binom{n-1}{m} = m\binom{n}{m+1} + \binom{n-1}{m+1}.\end{aligned}$$

Replacing all of this in (4), we get

$$\begin{aligned}d_n = &\left(\sum_{k=2}^{m}(k-1)\binom{n}{k}d_{n-k}\right) + m\binom{n}{m+1}d_{n-m-1} \\ &+ \binom{n-1}{m+1}d_{n-m-1} + (n-m-1)\binom{n-1}{m}d_{n-m-2},\end{aligned}$$

which completes the proof of (3). By considering the case $m = n$ in (3), we get (1).

Average number of people getting their coat back

Let $A(n)$ be the average number of people getting their coat back. Note that the permutations with precisely k people getting their coat back is $\binom{n}{k}d_{n-k}$. Therefore $A(n) = B(n)/n!$, where

$$B(n) = \sum_{k=1}^{n} k\binom{n}{k}d_{n-k}. \tag{6}$$

Therefore by using (1) in (6), we have

$$\begin{aligned}B(n) &= \sum_{k=1}^{n}\binom{n}{k}\cdot d_{n-k} + \sum_{k=1}^{n}(k-1)\cdot\binom{n}{k}\cdot d_{n-k} \\ &= \sum_{k=1}^{n}\binom{n}{k}\cdot d_{n-k} + d_n = \sum_{k=0}^{n}\binom{n}{k}\cdot d_{n-k} = n!.\end{aligned}$$

The last equality follows from partitioning the $n!$ ways of distributing n coats depending on how many people get their coat. For a given number k there are $\binom{n}{k}d_{n-k}$ ways of distributing the coats so that precisely k people get their coat.

So on average the number of people who got the right coat is

$$A(n) = \frac{B(n)}{n!} = 1.$$

Acknowledgment. The authors wish to thank the referees for their helpful suggestions.

Summary. We present a proof that the expected number of fixed points in a permutation is 1. The proof uses an unusual recursive formula for the number of derangements instead of the usual approaches with linearity of expectation or generating functions.

References

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