# ON EGYPTIAN FRACTIONS OF LENGTH 3

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Abstract. Let $a, n$ be positive integers that are relatively prime. We say that $a/n$ can be represented as an Egyptian fraction of length $k$ if there exist positive integers $m_1, \dots, m_k$ such that $\frac{a}{n} = \frac{1}{m_1} + \cdots + \frac{1}{m_k}$. Let $A_k(n)$ be the number of solutions $a$ to this equation. In this article, we give a formula for $A_2(p)$ and a parametrization for Egyptian fractions of length $3$, which allows us to give bounds to $A_3(n)$, to $f_a(n) = \#\{(m_1, m_2, m_3) : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\}$, and finally to $F(n) = \#\{(a, m_1, m_2, m_3) : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\}$.

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## 1. Introduction

***Historical background.*** The most ancient mathematical texts are mostly related to computations involving proportions, fractions, inverse of integers (sometimes in link with problems related to geometry). Many traces of these mathematics are found in Sumerian or Babylonian clay tablets covering a period of several millennia[1].

For Egyptian mathematics, many papyri present computations involving sums of unit fractions (fractions of the form $1/n$) and sometimes also the fraction $2/3$; see e.g. the Rhind Mathematical Papyrus. This document, estimated from 1550 BCE, is a copy by the scribe Ahmes of older documents. For example, it gives a list of decompositions of $2/n$ into unit fractions; such decompositions are also found in the Lahun Mathematical Papyri (UC $32159$ and UC $32160$, conserved at the University College London), which are dated circa 1800 BCE; see [27].

As traditional, we call an *Egyptian fraction decomposition* (or, in short, an Egyptian fraction) any rational number $a/n$, seen as a sum of unit fractions (obviously, all rational numbers possess such a decomposition!). It is often said that Egyptian fractions were related to parts of the Eye of Horus (an ancient Egyptian symbol of protection and royal power). However, this esoteric hypothesis (made popular via the seminal work of the Egyptologist Gardiner) is nowadays refuted [37].

As narrated in his survey [20], Ron Graham once asked André Weil what he thought to be the reason that led Egyptians to use this numerical system. André Weil answered jokingly "It is easy to explain. They took a wrong turn!". However, it is fair to say that, though it is not the most efficient system, it possesses interesting algorithmic aspects and has several applications: for a modern overview of the use of fractions in Egyptian mathematics, see [36].

Babylonian and Greek mathematics were later further developed by Arabic and Indian mathematicians. One book which played an important role in the transmission of Arabic mathematics to Europe is the *Liber Abbacci* of Fibonacci, in 1202 (see [33] for a translation into English). This book focuses mostly on the use of fractions

[1] In case the reader may have the chance to visit the corresponding museums, let us mention e.g. the Sumerian tablets from Shuruppak (Istanbul Museum, dated circa 2500 BCE), the Babylonian tablets VAT $6505$, $7535$, $7621$, $8512$ (Berlin Museum), Plimpton $322$ (Columbia University), $015-189$ (Hermitage Museum), YBC $4675$ (Yale University), AO $64456$, AO $17264$, and AO $6555$ (the Esagil tablet, Louvre Museum, dated 229 BCE), ...

and on the modus Indorum (the method of the Indians), i.e. the Hindu-Arabic numeral base 10 system that we all use nowadays. He shows how to use these two concepts to solve many problems, often related to trading/financial computations. With respect to fractions, he presents several methods to get Egyptian fraction decompositions, like e.g.

$$\frac{97}{100} = \frac{1}{50} + \frac{1}{5} + \frac{1}{4} + \frac{1}{2}.$$

Alternatively, a greedy method (nowadays called Fibonacci's greedy algorithm for Egyptian fractions) gives

$$\frac{97}{100} = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{86} + \frac{1}{25800}.$$

Similar decompositions were later also considered by Lambert [29] and Sylvester [42]. Sylvester's attention for this topic was in fact due to the father of the history of mathematics discipline, Moritz Cantor, who mentions (a few years after the translation of the Rhind papyrus) these Egyptian mathematics in the first volume of his monumental 4000-page *Vorlesungen über die Geschichte der Mathematik* [11].

***Modern times.*** Later, in the midst of the twentieth century, Erdős attracted mathematicians' attention to this topic, by proving or formulating puzzling conjectures related to Egyptian fraction decompositions, and also by establishing nice links with number theory. Egyptian fractions were e.g. the subject of the third published article of Erdős (the sum of unit fractions with denominators in arithmetic progression is not an integer [17]) and of his last (posthumous) published article with Graham and Butler (all integers are sums of unit fractions with denominators involving 3 distinct prime factors, [10]). Erdős also popularized some conjectures, analysed densities related to these fractions [16, 18, 40], and considered the minimal number of unit fractions needed to express a rational [6–8].

There are still many unsolved problems regarding Egyptian fractions; see e.g. [21, Section D.11] for a survey. We can end by mentioning further applications or links with total parallel resistance ($\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \dots$), trees and Huffman codes [23], Diophantine equations [1, 43], Engel expansion [12], continued fractions and Farey series [5, 22], products of Abelian groups [2], combinatorial number theory [19, 20, 34], and many asymptotic analyses [9, 13–15, 24–26, 28, 30].

***Our result.*** Our article analyzes the Egyptian fraction Diophantine equation

$$\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \text{ where } a, n, m_1, m_2, m_3 \in \mathbb{N}. \tag{1}$$

The famous Erdős–Straus conjecture asserts that, for $a = 4$, there is always a solution to this equation (for any $n > 1$); see [18] for the origin of this conjecture and see [31, Chapter 30.1] for some nontrivial progresses on it. A lesser-known conjecture due to Sierpiński asserts that, for $a = 5$, there is always a solution [41], and we additionally conjecture that this is also the case for $a = 6$ and $a = 7$ (for $n \geq a/3$). In fact, a conjecture of Schinzel [39] asserts that any positive integer $a$ is a solution of Equation (1) for $n$ large enough (e.g., it seems that $8/n$ is a sum of 3 unit fractions for $n > 241$). For sure, for each $n$, there is a finite number of integers $a$ which can be solution: the structure of the equation constrains $a$ to be between $1$ and $3n$. For fixed $n$, let $A_k(n) := \#\{a : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k}\}$. It is shown in [14] that $A_2(n) \ll n^{\epsilon}$ and that, for $k \geq 3$, one has

$$A_k(n) \ll n^{\alpha_k + \varepsilon}, \quad \text{where } \alpha_k = 1 - 2/(3^{k-2} + 1). \tag{2}$$

In particular, $A_3(n) \ll n^{1/2+\varepsilon}$. Here and in what follows, all implied constants in the Vinogradov symbol depend on a parameter $\varepsilon > 0$ which can be taken arbitrarily small. In this article, we give a different proof of $A_3(n) \ll n^{1/2+\varepsilon}$ and we get the following explicit inequality:

**Theorem 1.** *Introducing $h(n) := C/\log\log n$ (for some constant $C \approx 1.066$ given in Lemma 2 in Section 4), one has for $n \geq 57000$:*

$$A_3(n) \leq 10 n^{\frac{1}{2} + \frac{13}{4} h(n)} \log n.$$

In order to prove this result in Section 4, we give in Lemma 1 of Section 3 a parametrization of the solutions to (1). Furthermore, thanks to this parametrization lemma, in Section 5 we prove bounds on

$$f_a(n) := \#\left\{(m_1, m_2, m_3) : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\}$$

and

$$F(n) := \#\left\{(a, m_1, m_2, m_3) : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\}.$$

Note that $f_a(n)$ counts the number of representations of $a/n$ as an Egyptian fraction of length 3, while $F(n)$ counts all possible Egyptian fractions of length 3 with denominator $n$. We also include a formula and numerical tables in Section 2.

## 2. A formula and some numerics

In this section, we give a first few values of our main sequences. In addition to the sequences $A_k(n)$ which count the integers $a$ which are solutions of the Egyptian fraction Diophantine equation

$$\frac{a}{n} = \sum_{i=1}^{k} \frac{1}{m_i} \quad \text{for some positive integers } m_1, \ldots, m_k, \tag{3}$$

we shall also make use of some auxiliary sequences, $A_k^*(n)$, which consist of the number of integers $a$ which are solutions of Equation (3), with the additional constraint that $a$ is coprime to $n$.

The sequences $A_k(n)$ and $A_k^*(n)$ are easily computed via an exhaustive search. Some values can be more directly computed via the following closed-form formula.

**Proposition 1.** *If $p$ is prime, then $A_2(p) = 2 + d(p+1)$ and $A_2^*(p) = d(p+1)$, where $d(n) = \sum_{d|n} 1$ denotes as usual the number of divisors of $n$.*

*Proof.* First, if $a$ is any divisor of $n+1$, say $a = (n+1)/f$, then one has the decomposition $a/n = 1/(nf) + 1/f$. Let us now prove that all the decompositions are of this type, whenever $n = p$ is prime and $\gcd(a,n) = 1$. Equation (3) can be rewritten $am_1m_2 = n(m_1 + m_2)$. As $\gcd(a,n) = 1$, this is forcing $n|m_1m_2$. This gives that $m_1$ or $m_2$ is a multiple of $p$. Without loss of generality, say $m_1 = pf$. Thus one has $am_1m_2 = n(pf + m_2)$, i.e. $afm_2 = pf + m_2$, which implies $f|m_2$. Setting $m_2 = fg$ and simplifying, one gets $afg = (p+g)$, so $g|p$. As $p$ is prime, either one has $g = 1$, which leads to $af = p+1$ (and thus $a$ is any divisor of $p+1$), either one has $g = p$, which leads to $afp = 2p$ (and thus $a = 1$ or $a = 2$). Altogether, this gives $d(p+1)$ possible values for $a$, all actually leading to a legitimate Egyptian fraction decomposition of $a/p$. This proves $A_2^*(p) = d(p+1)$.

Now, consider Equation (3) with $n = p$ (where $p$ is prime) and $\gcd(a,p) \neq 1$. This gives exactly two additional decompositions: $\frac{a}{p} = \frac{1}{2} + \frac{1}{2}$ (for $a = p$) and $\frac{a}{p} = \frac{1}{1} + \frac{1}{1}$ (for $a = 2p$). Thus, one has $A_2(p) = 2 + A_2^*(p) = 2 + d(p+1)$. ☐

Unfortunately, there is no such simple formula for composite $n$. The obstruction comes from the fact that the factors of $n$ spread between $m_1$ and $m_2$ (like in the above proof) but this leads to a more intricate disjunction of cases too cumbersome to be captured by a simple formula.

| $n$ | $A_2(n)$ | $A_3(n)$ | $n$ | $A_2(n)$ | $A_3(n)$ | $n$ | $A_2(n)$ | $A_3(n)$ | $n$ | $A_2(n)$ | $A_3(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 26 | 15 | 36 | 51 | 20 | 58 | 76 | 30 | 84 |
| 2 | 4 | 6 | 27 | 18 | 41 | 52 | 27 | 68 | 77 | 25 | 77 |
| 3 | 5 | 8 | 28 | 23 | 49 | 53 | 10 | 36 | 78 | 39 | 101 |
| 4 | 7 | 11 | 29 | 10 | 27 | 54 | 35 | 82 | 79 | 12 | 46 |
| 5 | 6 | 11 | 30 | 29 | 58 | 55 | 24 | 66 | 80 | 49 | 118 |
| 6 | 10 | 16 | 31 | 8 | 28 | 56 | 36 | 85 | 81 | 28 | 81 |
| 7 | 6 | 13 | 32 | 23 | 51 | 57 | 21 | 62 | 82 | 18 | 62 |
| 8 | 11 | 19 | 33 | 18 | 44 | 58 | 18 | 54 | 83 | 14 | 52 |
| 9 | 10 | 19 | 34 | 17 | 42 | 59 | 14 | 41 | 84 | 60 | 139 |
| 10 | 12 | 22 | 35 | 20 | 49 | 60 | 51 | 109 | 85 | 22 | 79 |
| 11 | 8 | 16 | 36 | 34 | 69 | 61 | 6 | 33 | 86 | 19 | 65 |
| 12 | 17 | 29 | 37 | 6 | 28 | 62 | 18 | 57 | 87 | 25 | 79 |
| 13 | 6 | 18 | 38 | 17 | 45 | 63 | 33 | 86 | 88 | 39 | 106 |
| 14 | 13 | 26 | 39 | 20 | 51 | 64 | 32 | 82 | 89 | 14 | 49 |
| 15 | 14 | 29 | 40 | 33 | 71 | 65 | 22 | 69 | 90 | 58 | 138 |
| 16 | 16 | 31 | 41 | 10 | 31 | 66 | 36 | 89 | 91 | 20 | 80 |
| 17 | 8 | 21 | 42 | 34 | 74 | 67 | 8 | 40 | 92 | 29 | 89 |
| 18 | 20 | 38 | 43 | 8 | 32 | 68 | 30 | 80 | 93 | 21 | 77 |
| 19 | 8 | 22 | 44 | 25 | 61 | 69 | 25 | 71 | 94 | 21 | 70 |
| 20 | 21 | 41 | 45 | 28 | 69 | 70 | 39 | 98 | 95 | 24 | 83 |
| 21 | 17 | 37 | 46 | 17 | 48 | 71 | 14 | 44 | 96 | 59 | 143 |
| 22 | 14 | 32 | 47 | 12 | 36 | 72 | 54 | 121 | 97 | 8 | 47 |
| 23 | 10 | 25 | 48 | 41 | 87 | 73 | 6 | 38 | 98 | 32 | 98 |
| 24 | 27 | 51 | 49 | 14 | 48 | 74 | 17 | 59 | 99 | 36 | 107 |
| 25 | 12 | 33 | 50 | 27 | 67 | 75 | 33 | 91 | 100 | 48 | 128 |

TABLE 1. Number $A_k(n)$ of integers $a$ which are solutions of the Egyptian fraction Diophantine equation $\frac{a}{n} = \frac{1}{m_1} + \cdots + \frac{1}{m_k}$, for $k = 2, 3$, and $n = 1, \ldots, 100$. The sequences $A_2(n)$ and $A_3(n)$ are OEIS A308219 and OEIS A308221 in the On-Line Encyclopedia of Integer Sequences.

| $n$ | $A_2^*(n)$ | $A_3^*(n)$ | $n$ | $A_2^*(n)$ | $A_3^*(n)$ | $n$ | $A_2^*(n)$ | $A_3^*(n)$ | $n$ | $A_2^*(n)$ | $A_3^*(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 26 | 7 | 15 | 51 | 9 | 32 | 76 | 10 | 34 |
| 2 | 2 | 3 | 27 | 8 | 22 | 52 | 9 | 27 | 77 | 13 | 51 |
| 3 | 3 | 5 | 28 | 7 | 18 | 53 | 8 | 33 | 78 | 7 | 27 |
| 4 | 3 | 5 | 29 | 8 | 24 | 54 | 7 | 22 | 79 | 10 | 43 |
| 5 | 4 | 8 | 30 | 6 | 13 | 55 | 12 | 42 | 80 | 11 | 35 |
| 6 | 3 | 5 | 31 | 6 | 25 | 56 | 9 | 28 | 81 | 10 | 40 |
| 7 | 4 | 10 | 32 | 7 | 20 | 57 | 10 | 35 | 82 | 6 | 28 |
| 8 | 4 | 8 | 33 | 7 | 23 | 58 | 6 | 24 | 83 | 12 | 49 |
| 9 | 5 | 11 | 34 | 7 | 18 | 59 | 12 | 38 | 84 | 12 | 34 |
| 10 | 4 | 8 | 35 | 10 | 28 | 60 | 9 | 24 | 85 | 10 | 50 |
| 11 | 6 | 13 | 36 | 7 | 18 | 61 | 4 | 30 | 86 | 9 | 30 |
| 12 | 4 | 8 | 37 | 4 | 25 | 62 | 8 | 26 | 87 | 12 | 47 |
| 13 | 4 | 15 | 38 | 7 | 20 | 63 | 11 | 38 | 88 | 10 | 37 |
| 14 | 5 | 10 | 39 | 11 | 28 | 64 | 9 | 31 | 89 | 12 | 46 |
| 15 | 5 | 13 | 40 | 8 | 22 | 65 | 12 | 43 | 90 | 10 | 29 |
| 16 | 5 | 12 | 41 | 8 | 28 | 66 | 9 | 24 | 91 | 10 | 52 |
| 17 | 6 | 18 | 42 | 7 | 19 | 67 | 6 | 37 | 92 | 9 | 36 |
| 18 | 5 | 11 | 43 | 6 | 29 | 68 | 10 | 33 | 93 | 10 | 44 |
| 19 | 6 | 19 | 44 | 8 | 24 | 69 | 12 | 41 | 94 | 7 | 31 |
| 20 | 6 | 14 | 45 | 9 | 29 | 70 | 8 | 28 | 95 | 12 | 53 |
| 21 | 8 | 19 | 46 | 5 | 20 | 71 | 12 | 41 | 96 | 11 | 36 |
| 22 | 4 | 13 | 47 | 10 | 33 | 72 | 10 | 30 | 97 | 6 | 44 |
| 23 | 8 | 22 | 48 | 9 | 24 | 73 | 4 | 35 | 98 | 11 | 37 |
| 24 | 6 | 14 | 49 | 8 | 35 | 74 | 9 | 28 | 99 | 13 | 52 |
| 25 | 6 | 22 | 50 | 9 | 23 | 75 | 13 | 40 | 100 | 12 | 42 |

TABLE 2. Number $A_k^*(n)$ of integers $a$ which are solutions of the Egyptian fraction Diophantine equation $\frac{a}{n} = \frac{1}{m_1} + \cdots + \frac{1}{m_k}$ (with $a$ coprime to $n$), for $k = 2, 3$, and $n = 1, \ldots, 100$. The sequences $A_2^*(n)$ and $A_3^*(n)$ are OEIS A308220 and OEIS A308415 in the On-Line Encyclopedia of Integer Sequences.

## 3. A parametrization lemma

The proof of Theorem 1 is based on the following lemma which characterizes the solutions of Equation (4) below for $k = 3$. A similar (but simpler) characterization for $k = 2$ appears as Lemma 1 in [14] or in [3, 4, 35]; see also [38] for another existence criterion when $a = 4$.

**Lemma 1** (Parametrization lemma)**.** *Consider an Egyptian fraction decomposition of the irreducible fraction $a/n$:*

$$\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_k} \qquad (\text{with } \gcd(a,n) = 1 \text{ and } k = 3)^2. \tag{4}$$

*Then there exist $2k$ integers $D_1, \ldots, D_k, v_1, \ldots, v_k$ with*

(i) $\operatorname{lcm}[D_1, \ldots, D_k] \mid n$ *and* $\gcd(D_1, \ldots, D_k) = 1$*;*

(ii) $a v_1 \cdots v_k \mid \sum_{j=1}^{k} D_j v_j$ *and* $\gcd(v_i, D_j v_j) = 1$ *when* $i \neq j$*,*

*and the denominators of the Egyptian fractions are given by*

$$m_i = \frac{n \sum_{j=1}^{k} D_j v_j}{a D_i v_i} \qquad \text{for } i = 1, \ldots, k. \tag{5}$$

*Conversely, if conditions (i)–(ii) are fulfilled, then the $m_i$'s defined via (5) are integers, and denominators of $k$ unit fractions summing to $a/n$.*

**Remark 1.** This decomposition may not be unique. For example, both

$$(D_1, D_2, D_3, v_1, v_2, v_3) = (3, 1, 1, 3, 2, 1) \text{ and}$$
$$(D_1, D_2, D_3, v_1, v_2, v_3) = (9, 1, 1, 1, 2, 1)$$

correspond to the decomposition $\frac{2}{27} = \frac{1}{18} + \frac{1}{81} + \frac{1}{162}$.

**Remark 2.** It may be tempting to state the very same lemma for $k > 3$. However, this is not working: indeed, already for $k = 4$ one may have denominators $m_i$ such that there are no tuples of $D_k$'s, $v_k$'s satisfying (i) and (ii). This is e.g. the case for

$$\frac{1}{13} = \frac{1}{14} + \frac{1}{364} + \frac{1}{365} + \frac{1}{132860}.$$

Only the converse direction works for any $k$: if the tuples of $D_k$'s and $v_k$'s do exist, then they give a decomposition.

[2] Though Lemma 1 holds verbatim for $k = 3$ only, we state it with the parameter $k$ as we also discuss variations of this lemma for different values of $k$.

*Proof of Lemma 1.* Let $g = \gcd(m_1, m_2, m_3)$. Write $m_i = gm'_i$ for $i = 1, 2, 3$. So, $\gcd(m'_1, m'_2, m'_3) = 1$. We get

$$\frac{ag}{n} = \frac{1}{m'_1} + \frac{1}{m'_2} + \frac{1}{m'_3}.$$

Further, the left-hand side fraction gets irreducible by simplifying it via the factorizations

$$g = \gcd(g, n)g' \text{ and } n = \gcd(g, n)n',$$

so one obtains

$$\frac{ag'}{n'} = \frac{1}{m'_1} + \frac{1}{m'_2} + \frac{1}{m'_3}. \tag{6}$$

Put

$$P = \prod_{p \mid m'_1 m'_2 m'_3} p.$$

Note that no prime factor $p$ of $P$ divides all three of $m'_1, m'_2, m'_3$. Split them as follows:

- $Q$ is the largest divisor of $P$ formed with primes $p$ that divide just one of the $m'_1, m'_2, m'_3$.
- $R$ is the largest divisor of $P$ formed with primes $p$ which divide two of $m'_1, m'_2, m'_3$, say $m'_i$ and $m'_j$ but[3] $\nu_p(m'_i) \neq \nu_p(m'_j)$.
- $S = P/(QR)$ (i.e., the product of the remaining primes, those having the same valuation in two of the $m_i$'s).

For $i = 1, 2, 3$, write

$$m'_i = q_i r_i s_i, \tag{7}$$

where $q_i$ is formed only of primes from $Q$, $r_i$ is formed of primes from $R$, and $s_i$ is formed of primes from $S$. We show that

$$q_1 q_2 q_3 \operatorname{lcm}[r_1, r_2, r_3] \mid n'; \tag{8}$$

$$s_1 = u_2 u_3, \; s_2 = u_1 u_3, \; s_3 = u_1 u_2 \quad \text{for some integers } u_1, u_2, u_3. \tag{9}$$

[3]We use the classical notation $\nu_p(m)$ for the exponent of $p$ in the factorization of $m$.

Now, rewrite (6) as

$$\frac{ag'}{n'} = \frac{1}{q_1r_1s_1} + \frac{m_2'+m_3'}{m_2'm_3'} = \frac{m_2'm_3' + q_1r_1s_1(m_2'+m_3')}{q_1r_1s_1m_2'm_3'},$$

In the right-hand side, $q_1$ is coprime to $m_2'm_3' + q_1r_1s_1(m_2'+m_3')$ (because by the definition of $Q$, $q_1$ is coprime to $m_2'm_3'$). So, it must be the case that $q_1 \mid n'$, as $ag'/n'$ is irreducible. Similarly, $q_2, q_3$ divide $n'$ and since any two of the $q_i$'s are mutually coprime, it follows that $q_1q_2q_3 \mid n'$. Consider next $r_1$. It is formed by primes from $R$, so for each prime factor $p$ of $r_1$ there exists $i \in \{2,3\}$ such that $p \mid r_i$. Say $i = 2$, then we introduce $\alpha_1 := \nu_p(r_1)$ and $\alpha_2 := \nu_p(r_2)$, with $\alpha_2 > \alpha_1$ (these two assumptions $i = 2$, and $\alpha_i > \alpha_1$, are without loss of generality of this proof: the other cases would be handled similarly). Now, writing $m_1' = p^{\alpha_1}m_1''$, $m_2' = p^{\alpha_2}m_2''$, we have

$$\frac{ag'}{n'} = \frac{1}{p^{\alpha_1}m_1''} + \frac{1}{p^{\alpha_2}m_2''} + \frac{1}{m_3'} = \frac{m_3'm_2''p^{\alpha_2-\alpha_1} + m_1''m_3' + p^{\alpha_2}m_2''m_1''}{p^{\alpha_2}m_1''m_2''m_3'}.$$

In the right, $p^{\alpha_2}$ is coprime to the numerator $m_3'm_2''p^{\alpha_2-\alpha_1} + m_1''m_3' + p^{\alpha_2}m_2''m_1''$. Thus, $p^{\alpha_2} \mid n'$. Note that $p^{\alpha_2} = \mathrm{lcm}[p^{\alpha_1}, p^{\alpha_2}]$. Proceeding one prime at time for the primes dividing $r_1, r_2, r_3$, we get to the conclusion that $\mathrm{lcm}[r_1, r_2, r_3] \mid n'$. Since $q_1q_2q_3$ and $\mathrm{lcm}[r_1, r_2, r_3]$ have no prime factor in common, and since $q_1q_2q_3 \mid n'$, this proves Formula (8).

Now, Formula (9) is a simple linear algebra problem. Namely, for $i_1 \in \{1,2,3\}$, let $i_2, i_3$ such that $\{1,2,3\} = \{i_1, i_2, i_3\}$ and write

$$s_{i_1} = s_{i_1}^{(i_2)} s_{i_1}^{(i_3)}, \qquad \text{where} \qquad s_{i_1}^{(i_2)} = \prod_{p \mid \gcd(s_{i_1}, s_{i_2})} p^{\nu_p(s_{i_1})}.$$

The condition that $\nu_p(s_{i_1}) = \nu_p(s_{i_2})$ if $p \mid \gcd(s_{i_1}, s_{i_2})$ shows that $s_{i_1}^{(i_2)} = s_{i_2}^{(i_1)}$ for all $i_1 \neq i_2$. This gives Formula (9).

Now, rewrite (6) using (7) and (8): putting $n'' = n'/(q_1q_2q_3\mathrm{lcm}[r_1, r_2, r_3])$, we get

$$\frac{ag'}{n''} = \frac{q_2q_3\frac{\mathrm{lcm}[r_1,r_2,r_3]}{r_1}u_1 + q_1q_3\frac{\mathrm{lcm}[r_1,r_2,r_3]}{r_2}u_2 + q_1q_2\frac{\mathrm{lcm}[r_1,r_2,r_3]}{r_3}u_3}{u_1u_2u_3}.$$

It is clear that $u_1, u_2, u_3$ are mutually coprime since any common prime factor of two of them will divide all three of $m_1', m_2', m_3'$. So, write $u_i = d_iu_i'$, where $d_i$ is the largest factor of $u_i$ whose prime factors divide $n''$ and $u_i$, and where $u_i'$ is coprime to $n''$. Similarly, write $n'' = d_1' d_2' d_3' n'''$, where $d_i'$ is the largest factor of $n''$ whose prime factors divide $d_i$. We then get

$$\frac{ag'}{n'''} \prod_{j=1}^{3} \left( \frac{d_j}{d_j'} \right) = \frac{q_2 q_3 \frac{\mathrm{lcm}[r_1,r_2,r_3]}{r_1} u_1 + q_1 q_3 \frac{\mathrm{lcm}[r_1,r_2,r_3]}{r_2} u_2 + q_1 q_2 \frac{\mathrm{lcm}[r_1,r_2,r_3]}{r_3} u_3}{u_1' u_2' u_3'}. \tag{10}$$

In the right, $u_1' u_2' u_3'$ is divisible only by primes coprime to $n''$ so $u_1' u_2' u_3'$ divides the numerator

$$q_2 q_3 \frac{\mathrm{lcm}[r_1, r_2, r_3]}{r_1} u_1 + q_1 q_3 \frac{\mathrm{lcm}[r_1, r_2, r_3]}{r_2} u_2 + q_1 q_2 \frac{\mathrm{lcm}[r_1, r_2, r_3]}{r_3} u_3.$$

So, the left-hand side of (10) is an integer. This shows that $d_i' \mid d_i$ for $i = 1, 2, 3$ and $n''' = 1$ (since the four quantities $d_i/d_i'$ for $i = 1, 2, 3$ and $n'''$ are rational numbers supported on mutually disjoint sets of prime factors of $n''$ and $ag'$ is coprime to $n''$). Thus, in fact $n'' = d_1' d_2' d_3'$ and we can write $u_i = d_i' v_i$, where $v_i = (d_i/d_i') u_i'$. Hence, we get

$$ag' = \frac{q_2 q_3 \frac{\mathrm{lcm}[r_1,r_2,r_3]}{r_1} d_1' v_1 + q_1 q_3 \frac{\mathrm{lcm}[r_1,r_2,r_3]}{r_2} d_2' v_2 + q_1 q_2 \frac{\mathrm{lcm}[r_1,r_2,r_3]}{r_3} d_3' v_3}{v_1 v_2 v_3}. \tag{11}$$

Putting (for $i \in \{1, 2, 3\}$)

$$D_i := \frac{q_1 \cdots q_3}{q_i} \frac{\mathrm{lcm}[r_1, r_2, r_3]}{r_i} d_i',$$

we have that each $D_i$ is a divisor of $n' = n/\gcd(g, n)$, so

$$\mathrm{lcm}[D_1, D_2, D_3] \mid q_1 q_2 q_3 \mathrm{lcm}[r_1, r_2, r_3] d_1' d_2' d_3' = n' d_1' d_2' d_3' = n' n'' = n,$$

which is part of condition (i) of our parametrization lemma (Lemma 1). The second part of condition (i) is now easy. Indeed, $\gcd(D_1, D_2, D_3)$ cannot be divisible by primes from either $Q$ or $R$, and $d_i'$ is coprime to $d_j'$ (since $d_i'$ and $d_j'$ are supported on primes dividing $d_i$ and $d_j$ which are divisors of $u_i$ and $u_j$, respectively), which shows that indeed $\gcd(D_1, D_2, D_3) = 1$. Rewriting Equation (11) in terms of these $D_i$'s gives

$$a v_1 v_2 v_3 \mid D_1 v_1 + D_2 v_2 + D_3 v_3,$$

which is the first part of condition (ii) of our parametrization lemma (Lemma 1). The second part is also clear since $v_i$ is a divisor of $u_i$, which is coprime to $u_j$ for any $j \neq i$ with $\{i, j\} \subset \{1, 2, 3\}$. The converse direction is obvious: if one has the divisibility conditions (i)–(ii), it is clear that the $m_i$'s defined via (5) are integers, and satisfy Equation (4). □

## 4. An explicit bound on $A_3(n)$.

To prove explicit results, we will use the following lemma from [32].

**Lemma 2** (Nicolas-Robin). *Let* $d(n)$ *be the number of divisors of* $n$ *and let* $h(n) := C/\log\log n$, *where* $C := \frac{2\log(48)\log(\log(6983776800))}{\log(6983776800)} \approx 1.066$. *Then*

$$d(n) \le n^{h(n)}.$$

Because of this lemma, we will use $h(n) = C/\log\log n$ for the rest of the paper.

*Proof of Theorem 1.* Consider

$$\mathcal{A}_3^*(n) = \left\{ a \,:\, \gcd(a,n) = 1,\ \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right\} \quad \text{and} \quad A_3^*(n) = \#\mathcal{A}_3^*(n).$$

From the parametrization lemma (Lemma 1), if $a \in \mathcal{A}_3^*(n)$, there exist integers $D_1, D_2, D_3, v_1, v_2, v_3$ satisfying $D_i | n$, $v_1v_2v_3 | D_1v_1 + D_2v_2 + D_3v_3$, and

$$a \,\Big|\, \frac{D_1v_1 + D_2v_2 + D_3v_3}{v_1v_2v_3}.$$

Let $A$ be such that $Av_1v_2v_3 = D_1v_1 + D_2v_2 + D_3v_3$. Then $a|A$.

First suppose $A \le n^{1/2+\alpha}$. Then $A_3^*(n)$ is bounded above by

$$\sum_{A \le n^{1/2+\alpha}} d(A) \le n^{1/2+\alpha} \log\left(n^{1/2+\alpha}\right) + n^{1/2+\alpha}$$

$$= \left(\frac{1}{2} + \alpha\right) n^{1/2+\alpha} \log n + n^{1/2+\alpha}.$$

Now, suppose $A > n^{1/2+\alpha}$. Fix $D_1, D_2, D_3$ as divisors of $n$. There are $d(n)^3 \le n^{3h(n)}$ ways of doing this. Suppose $v_1, v_2 \le v_3$; one then has

CB: 3 possibilities, to rephrase

$$Av_1v_2v_3 = D_1v_1 + D_2v_2 + D_3v_3 \le (D_1 + D_2 + D_3)v_3 \le 3nv_3.$$

Therefore $m := v_1v_2 \le 3n^{1/2-\alpha}$. Once $v_1, v_2$ are chosen, there are at most $d(D_1v_1 + D_2v_2)$ choices of $v_3$ (since $v_1v_2v_3 | D_1v_1 + D_2v_2 + D_3v_3$). We have

$$D_1v_1 + D_2v_2 \le 2nv_3 \le 6n^{3/2-\alpha}.$$

Therefore

$$d(D_1v_1 + D_2v_2) \le 6^{h(n)} n^{3h(n)/2 - \alpha h(n)}.$$

We can thus bound the contribution of the $a$'s appearing when $A > n^{1/2+\alpha}$ by

$$3n^{3h(n)} 6^{h(n)} n^{3h(n)/2 - \alpha h(n)} \sum_{m \le 3n^{1/2-\alpha}} d(m).$$

Given that

$$\sum_{m\le 3n^{1/2-\alpha}} d(m) \le 3n^{\frac{1}{2}-\alpha}\log(3n^{\frac{1}{2}-\alpha})+3n^{\frac{1}{2}-\alpha} = 3n^{\frac{1}{2}-\alpha}\left(\frac{1}{2}-\alpha\right)\log n+3n^{\frac{1}{2}-\alpha}\log(3e),$$

and that $6^{h(n)} \le \frac{20}{9}$ for $n \ge 57000$, we get

$$\begin{aligned} A_3^*(n) \le & 10n^{\frac{1}{2}+\frac{9}{2}h(n)-\alpha-\alpha h(n)}\log n + 20n^{\frac{1}{2}+\frac{9}{2}h(n)-\alpha-\alpha h(n)}\log(3e) \\ & - 20n^{\frac{1}{2}+\frac{9}{2}h(n)-\alpha-\alpha h(n)}\log n + \left(\frac{1}{2}+\alpha\right)n^{\frac{1}{2}+\alpha} + n^{\frac{1}{2}+\alpha}. \end{aligned}$$

Choose $\alpha = \frac{9}{4+2h(n)}h(n) \le \frac{9}{4}h(n)$. We then have

$$\begin{aligned} A_3^*(n) &\le n^{\frac{1}{2}+\frac{9}{4}h(n)}\left(10\log n + 20\log(3e) - 20\alpha\log n + \frac{1}{2}+\alpha+1\right) \\ &\le 10n^{\frac{1}{2}+\frac{9}{4}h(n)}\log n. \end{aligned}$$

For the last inequality we use that for $n \ge 20$, $h(n) \le 1$, so

$$\alpha \ge \frac{9}{2+2h(n)}h(n) \ge \frac{9}{4}h(n), \quad \text{and} \quad \alpha \le \frac{9}{2}h(n) \le \frac{9}{2}.$$

For $n > e$, $\log n/\log\log n \ge e$, therefore

$$20\alpha\log n \ge 45h(n)\log n > 45e\cdot C > 20\log(3e) + \frac{3}{2}+\frac{9}{2}.$$

Therefore, for $n \ge 57000$,

$$A_3^*(n) \le 10n^{\frac{1}{2}+\frac{9}{4}h(n)}\log n.$$

This gives the statement of Theorem 1:

$$A_3(n) = \sum_{d|n} A_3^*(d) \le 10n^{\frac{1}{2}+\frac{13}{4}h(n)}\log n. \qquad \square$$

**Corollary 1.** *For* $n \ge 10^{10^{23}}$,

$$A_3(n) < \frac{1}{100}n^{\frac{1}{2}+\frac{1}{15}}.$$

*Proof.* When $n \ge 10^{10^{23}}$,

$$\frac{13}{4}h(n) + \frac{\log\log n}{\log n} + \frac{\log 1000}{\log n} < \frac{1}{15}.$$

Therefore, using Theorem 1, we get

$$A_3(n) \le 10n^{\frac{1}{2}+\frac{13}{4}h(n)}\log n = \frac{1}{100}n^{\frac{1}{2}+\frac{13}{4}h(n)+\frac{\log\log n}{\log n}+\frac{\log 1000}{\log n}} < \frac{1}{100}n^{\frac{1}{2}+\frac{1}{15}}. \qquad \square$$

## 5. On the number of length 3 representations

In this section we study

$$f_a(n) = \#\left\{(m_1, m_2, m_3) : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\},$$

and

$$F(n) = \#\left\{(a, m_1, m_2, m_3) : \frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right\}.$$

**Theorem 2.**

$$f_a(n) \le n^{\varepsilon}\left(\frac{n^{1/2+\rho/2}}{a} + n^{1-\rho}\right). \tag{12}$$

Choosing $\rho = 1/3 + (2/3)(\log a/\log n)$ to balance between the two estimates, we get that

$$f_a(n) \ll \frac{n^{2/3+\varepsilon}}{a^{2/3}}.$$

In particular, $f_a(n) \ll n^{2/3+\varepsilon}$ uniformly in $a$. In [15, Proposition 1.7], it is shown that for primes $p$ one has as $p \to \infty$

$$f_4(p) \ll p^{3/5+o(1)}.$$

The argument from [15] applies to the case when $n$ is replaced by a composite integer but only solutions of a certain kind are counted (in our notation for which $\{D_1, D_2, D_3\} \subseteq \{1, n\}$ which are the solutions that "look like" the solutions for the primes by changing $p$ to $n$ wherever we see it in the two cases), whereas we count all solutions and for all $a$ and all $n$. Our argument is slightly worse (it gives the exponent $2/3 + \varepsilon$) but it works for fixed or bounded $a$ and it allows us to get better exponents for larger $a$ of size $n^c$ for some positive constant $c$.

*Proof of Theorem 2.* For the proof, we use the parametrization lemma (Lemma 1). Indeed, there are divisors $D_1, D_2, D_3$ of $n$ such that

$$a \mid (D_1 v_1 + D_2 v_2 + D_3 v_3)/(v_1 v_2 v_3).$$

Fix $D_1, D_2, D_3$. They can be fixed in at most $d(n)^3 \ll n^{\varepsilon}$ ways. Assume $v_1 \le v_2 \le v_3$. Furthermore, we replace $a$ by $A := ab = (D_1 v_1 + D_2 v_2 + D_3 v_3)/(v_1 v_2 v_3)$, which is an integer. Note that

$$\frac{a}{n} = \frac{1}{b(n/D_1)v_2 v_3} + \frac{1}{b(n/D_2)v_1 v_3} + \frac{1}{b(n/D_3)v_1 v_2} := \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}$$

and that

$$Av_1v_2v_3 = D_1v_1 + D_2v_2 + D_3v_3 \le 3nv_3.$$

Now, let $\rho$ be a parameter to be fixed later. First, let us assume that $A > n^\rho$. Then $v_1v_2 \le 3n^{1-\rho}$, so the pair $(v_1, v_2)$ can be chosen in $n^{1-\rho+\varepsilon}$ ways. Having chosen $(v_1, v_2)$, $v_3$ is a divisor of $D_1v_1 + D_2v_2$, so it can be chosen in $n^\varepsilon$ ways, and after that everything is determined, so $A$ is unique. Note that such $A$ might not end up being divisible with the number $a$ we started with so not all such solutions will contribute to $f_a(n)$. This gives the second part of the right–hand side inequality in the statement of the theorem. So, we may assume that $ab \le n^\rho$, so

$$b \le \frac{n^\rho}{a}, \tag{13}$$

and then we have $v_1 \le 3(n/ab)^{1/2}$. Fix $v_1$. It can be fixed in at most $3(n/ab)^{1/2}$ ways. Now put $A_1 := Av_1 = (ab)v_1,\ B_1 := D_1v_1$ and note that they are fixed. Further

$$A_1v_2v_3 = B_1 + D_2v_2 + D_3v_3$$

and the only variables are $v_2, v_3$. The above can be rewritten as

$$A_1v_2v_3 - D_2v_2 - D_3v_3 + D_3D_2/A_1 = B_1 + (D_2D_3/A_1);$$

or equivalently as

$$(A_1v_3 - D_2)(A_1v_2 - D_3) = A_1B_1 + D_2D_3.$$

It thus follows that $A_1v_2 - D_3$ can be chosen in $d(A_1B_1 + D_2D_3) \ll n^\varepsilon$ ways and then $v_3$ is uniquely determined. We thus get that for fixed $b$, $v_1$, $D_1, D_2, D_3$ there are $n^\varepsilon$ possibilities for $(v_2, v_3)$. Summing up over $v_1$, it follows that there are

$$\ll n^\varepsilon (n/ab)^{1/2}$$

possibilities. Summing over $b \le n^\rho/a$, we get a count of

$$\frac{n^{1/2+\varepsilon}}{a^{1/2}} \sum_{b \le 3n^\rho/a} \frac{1}{b^{1/2}} \ll \frac{n^{1/2+2\varepsilon}}{a^{1/2}} \int_1^{3n^\rho/a} \frac{dt}{t^{1/2}} \ll \frac{n^{1/2+\rho/2+2\varepsilon}}{a}. \tag{14}$$

Thus,

$$f_a(n) \le n^\varepsilon \left( \frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right),$$

which is (12). □

**Theorem 3.** *Let* $F(n) = \sum_a f_a(n)$ *be the count of all* $(a, m_1, m_2, m_3)$ *such that* $\frac{a}{n} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}$. *We have* $F(n) \ll n^{5/6+\varepsilon}$.

*Proof.* Let $\varepsilon > 0$. Note

$$\sum_{a \le n^\alpha} f_a(n) \le n^{2/3+\alpha+\varepsilon}. \tag{15}$$

This estimate follows from using $\rho = 1/3$ in

$$f_a(n) \le n^\varepsilon \left( \frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right) \le n^\varepsilon \left( n^{1/2+\rho/2} + n^{1-\rho} \right).$$

Now note

$$\sum_{n^\alpha \le a \le n^\beta} f_a(n) \le n^{2/3+\frac{3\beta-2\alpha}{3}+\varepsilon}. \tag{16}$$

This follows from using $\rho = \frac{2\alpha+1}{3}$ in

$$f_a(n) \le n^\varepsilon \left( \frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right) \le n^\varepsilon \left( n^{1/2+\rho/2-\alpha} + n^{1-\rho} \right).$$

Finally note

$$\sum_{a \ge n^\gamma} f_a(n) \le n^{1/2+\frac{2-2\gamma}{3}+\varepsilon}. \tag{17}$$

This follows from using $\rho = \frac{2\gamma+1}{3}$ and that $A_3(n) \ll n^{1/2+\varepsilon}$ in

$$f_a(n) \le n^\varepsilon \left( \frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right) \le n^\varepsilon \left( n^{1/2+\rho/2-\gamma} + n^{1-\rho} \right).$$

Now let $x_1 = 1/6$ and $x_k = \frac{1}{6} + \frac{2}{3}x_{k-1}$ for $k \ge 2$. Then $x_k = \frac{1}{2} - \frac{(2/3)^k}{2}$. Let $i$ be fixed such that $(2/3)^i < \varepsilon$. Consider the intervals

$$[1, n^{x_1}], [n^{x_1}, n^{x_2}], \ldots, [n^{x_{i-1}}, n^{x_i}], [n^{x_i}, n^{1/2}], [n^{1/2}, \infty).$$

From (15), (16), and (17), we have that

$$\sum_{a \in I} f_a(n) \ll n^{5/6+\varepsilon},$$

for any interval $I \ne [n^{x_i}, n^{1/2}]$. Using (16) and our choice of $i$, we get

$$\sum_{n^{x^i} \le a \le n^{1/2}} f_a(n) \ll n^{5/6+\frac{4}{3}\varepsilon}.$$

Therefore

$$F(n) \le (i+1)n^{5/6+\varepsilon} + n^{5/6+\frac{4}{3}\varepsilon} \le n^{5/6+2\varepsilon}.$$ □

**Problem 1.** *The first values for which $F(n) < n$ are: $F(8821) = 8590, F(11161) = 10270, F(11941) = 10120$. It is an open problem to find the largest $n$ such that $F(n) > n$. We can however show that such an $n$ is smaller than $10^{10^{23}}$.*

**Theorem 4.** *For $n \geq 10^{10^{23}}$, $F(n) < \frac{1}{10}n$.*

The proof of the theorem requires the explicit upper bound for $A_3(n)$ from Corollary 1. It also requires the following explicit version of Theorem 2:

**Theorem 5.** *Let $1/3 \leq \rho$, and $n \geq 11000$. Then*

$$f_a(n) \leq 6n^{5h(n)} \left( 6\sqrt{2}\, \frac{n^{1/2+\rho/2}}{a} 10^{h(n)} + \frac{3}{2}\, n^{1-\rho} \, \log n \, 6^{h(n)} \right). \tag{18}$$

*Proof.* For the proof, we use the parametrization lemma (Lemma 1). Therefore, there are divisors $D_1, D_2, D_3$ of $n$ such that

$$a \mid (D_1v_1 + D_2v_2 + D_3v_3)/(v_1v_2v_3).$$

Fix $D_1, D_2, D_3$. They can be fixed in at most $d(n)^3 \leq n^{3h(n)}$ (using Lemma 2). Introduce a factor of 6 by assuming $v_1 \leq v_2 \leq v_3$. Furthermore, we replace $a$ by $A := ab = (D_1v_1 + D_2v_2 + D_3v_3)/(v_1v_2v_3)$, which is an integer. Note that

$$\frac{a}{n} = \frac{1}{b(n/D_1)v_2v_3} + \frac{1}{b(n/D_2)v_1v_3} + \frac{1}{b(n/D_3)v_1v_2} := \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}.$$

Now,

$$Av_1v_2v_3 = D_1v_1 + D_2v_2 + D_3v_3 \leq 3nv_3.$$

First let us assume that $A > n^\rho$. Then $v_1v_2 \leq 3n^{1-\rho}$, so the pair $(v_1, v_2)$ can be chosen in at most

$$\begin{aligned}
\sum_{v_1 \leq \sqrt{3n^{1-\rho}}} \sum_{v_2 \leq 3n^{1-\rho}/v_1} 1 &\leq \sum_{v_1 \leq \sqrt{3n^{1-\rho}}} \frac{3n^{1-\rho}}{v_1} \\
&\leq 3n^{1-\rho} \log(\sqrt{3}n^{1/2-\rho/2}) + 3n^{1-\rho} \\
&\leq \frac{3}{2}n^{1-\rho} \log n - \frac{3\rho}{2}n^{1-\rho} \log n + \frac{3}{2}n^{1-\rho} \log 3 + 3n^{1-\rho} \\
&\leq \frac{3}{2}n^{1-\rho} \log n
\end{aligned}$$

ways. The last step of the inequality follows from using that $\rho \geq 1/3$ and $n \geq 11000$.

Having chosen $(v_1, v_2)$, $v_3$ is a divisor of $D_1v_1 + D_2v_2$, so it can be chosen in $d(D_1v_1+D_2v_2)$ ways, and after that everything is determined, so $A$ is unique. Now

$$D_1v_1 + D_2v_2 \leq (D_1 + D_2)v_2 \leq (2n)(3n) = 6n^2,$$

so $d(D_1v_1 + D_2v_2) \leq (6n^2)^{h(n)}$. This gives the second part of the right–hand side inequality in the statement of the theorem (after factoring out an $n^{2h(n)}$).

For the first part, we may assume that $ab \leq n^\rho$, so

$$b \leq \frac{n^\rho}{a}, \tag{19}$$

and then we have $v_1 \leq 3(n/ab)^{1/2}$. Fix $v_1$. It can be fixed in at most $3(n/ab)^{1/2}$ ways. Now put $A_1 := Av_1 = (ab)v_1,\ B_1 := D_1v_1$ and note that they are fixed. Further

$$A_1v_2v_3 = B_1 + D_2v_2 + D_3v_3$$

and the only variables are $v_2, v_3$. The above can be rewritten as

$$A_1v_2v_3 - D_2v_2 - D_3v_3 + D_3D_2/A_1 = B_1 + (D_2D_3/A_1);$$

or equivalently as

$$(A_1v_3 - D_2)(A_1v_2 - D_3) = A_1B_1 + D_2D_3.$$

It thus follows that $A_1v_2 - D_3$ can be chosen in $d(A_1B_1 + D_2D_3)$ ways and then $v_3$ is uniquely determined. Since

$$A_1B_1 + D_2D_3 = abD_1v_1^2 + D_2D_3 \leq abn\left(9\frac{n}{ab}\right) + n^2 = 10n^2,$$

then

$$d(A_1B_1 + D_2D_3) \leq (10n^2)^{h(n)}.$$

We have thus bounded the number of possibilities for $(v_2, v_3)$ given fixed $b$, $v_1$, $D_1, D_2, D_3$. To finish our estimate we use that $v_1 \leq 3(n/ab)^{1/2}$ and that

$$\sum_{b\leq n^\rho/a} \frac{1}{b^{1/2}} \leq \int_1^{\frac{n^\rho}{a}+1} \frac{1}{t^{1/2}}\,\mathrm{d}t \leq \int_1^{2\frac{n^\rho}{a}} \frac{1}{t^{1/2}}\,\mathrm{d}t \leq 2\sqrt{2}\frac{n^{\rho/2}}{a^{1/2}}. \qquad \square$$

**Corollary 2.** *If* $n \geq 10^{10^{23}}$, *then*

$$f_a(n) < \frac{1}{100} n^{\frac{1}{10}} \left( \frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right).$$

*Proof.* For $n \geq 10^6$, $6\sqrt{2} \cdot 10^{h(n)} \leq 2\log n$, and for $n \geq 10^{334}$, $\frac{3}{2} 6^{h(n)} \leq 2$. Therefore

$$f_a(n) \leq 12(\log n) n^{5h(n)} \left( \frac{n^{1/2+\rho/2}}{a} + n^{1-\rho} \right).$$

But we have, for $n \geq 10^{10^{23}}$,

$$12(\log n) n^{5h(n)} = \frac{1}{100} n^{5h(n) + \frac{\log\log n}{\log n} + \frac{\log 1200}{\log n}} < \frac{1}{100} n^{1/10}. \qquad \square$$

We are now ready to prove Theorem 4.

***Proof of Theorem 4.*** Let $n \geq 10^{10^{23}}$. The proof will be similar to the proof of Theorem 3. Applying Corollary 2 with $\rho = 1/3$ yields

$$\sum_{a \leq n^{7/30}} f_a(n) \leq \frac{2}{100} n^{\frac{2}{3} + \frac{7}{30} + \frac{1}{10}} = \frac{2}{100} n. \tag{20}$$

Now applying $\rho = 22/45$ to Corollary 2, we get

$$\sum_{n^{7/30} \leq a \leq n^{7/18}} f_a(n) \leq \frac{2}{100} n^{\frac{23}{45} + \frac{7}{18} + \frac{1}{10}} = \frac{2}{100} n. \tag{21}$$

Applying $\rho = 89/135$ to Corollary 2 yields

$$\sum_{n^{7/18} \leq a \leq n^{1/2}} f_a(n) \leq \frac{2}{100} n^{\frac{46}{135} + \frac{1}{2} + \frac{1}{10}} = \frac{2}{100} n^{\frac{127}{135}} < \frac{2}{100} n. \tag{22}$$

Applying $\rho = \frac{2}{3}$ to Corollary 2 and using Corollary 1 yields

$$\sum_{a \geq n^{1/2}} f_a(n) < \frac{2}{10000} n^{\frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{15}} = \frac{2}{10000} n. \tag{23}$$

The proof follows from combining (20), (21), (22), (23). $\square$

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