In Facebook, if you average the average number of friends of everybody (which as of 2012, it was around 635 according to [2]), it is much bigger than the average number of friends of individual users (which was 190 as of 2012 according to [2]). In fact about 93% (see [2]) of people have less friends than the average number of friends their friends have. This sort of phenomenon happens in any network, not just on Facebook\(^1\).

Let’s set up the notation for our proof of this theorem. Let \( G \) be a finite graph with \( n \) vertices. For a vertex \( v \in G \), we say that \( u \in E(v) \) if \( u \) is a vertex of \( G \) adjacent to \( v \) (i.e., if \( u \) and \( v \) are “friends”). Let \( d(v) \) be the degree of \( v \).

The average number of “friends” is

\[
\frac{1}{n} \sum_{v \in G} d(v).
\]

The average of the average number of friends is

\[
\frac{1}{n} \sum_{v \in G} \frac{1}{d(v)} \sum_{u \in E(v)} d(u).
\]

We’re ready to state the theorem (and prove it)

**Theorem 1.**

\[
\frac{1}{n} \sum_{v \in G} \frac{1}{d(v)} \sum_{u \in E(v)} d(u) \geq \frac{1}{n} \sum_{v \in G} d(v).
\]

**Proof.** Consider

\[
A = \frac{1}{n} \sum_{v \in G} \frac{1}{d(v)} \sum_{u \in E(v)} d(u).
\]

By changing the order of summation we get

\[
\frac{1}{n} \sum_{u \in G} d(u) \sum_{v \in E(u)} \frac{1}{d(v)}.
\]

By making a change of variable we also get

\[
\frac{1}{n} \sum_{v \in G} \sum_{u \in E(v)} \frac{d(v)}{d(u)}.
\]

\(^1\)For example, on Twitter, the percentage of users with less friends than the average number of friends their friends have is over 98\% according to [1] (as of 2009).
Therefore
\[ \frac{1}{n} \sum_{v \in G} \frac{1}{d(v)} \sum_{u \in E(v)} d(u) = \frac{1}{n} \sum_{v \in G} \sum_{u \in E(v)} \frac{d(v)}{d(u)}. \]

Since they are equal, their average is also $A$. Therefore, using that $x + \frac{1}{x} \geq 2$ for all $x > 0$ (the AM-GM inequality), we get
\[
A = \frac{1}{n} \sum_{v \in G} \sum_{u \in E(v)} \frac{d(v)}{d(u)} + \frac{d(u)}{d(v)} \geq \frac{1}{n} \sum_{v \in G} \sum_{u \in E(v)} 1 \geq \frac{1}{n} \sum_{v \in G} d(v).
\]

Which is what we wanted to prove.

\[\square\]

References
