Final Practice Exam
SOLUTIONS

(1)

x	y	$\neg y$	$x \to \neg y$	$x \to y$	$\neg(x \to y)$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	T	T	F
F	F	T	T	T	F

They are not equivalent.

(If $x = F$ $x \to \neg y$ is True while $\neg(x \to y)$ is FALSE)

(2)

a) if $x+1=1$, $x+1$ is positive while $x=0$ is not positive

(note $x=0$ is the only counterexample, if you pick x to be negative then x is negative and $x+1$ is not positive. If $x>0$ then both x and $x+1$ are positive).

b) 131 is a palindrome that's not divisible by 11.

Note: If the palindrome has an even number of digits, then it must be a multiple of eleven but if it has an odd number of digits then ~~only~~ less than $\frac{1}{10}$ of those palindromes are multiples of 11.

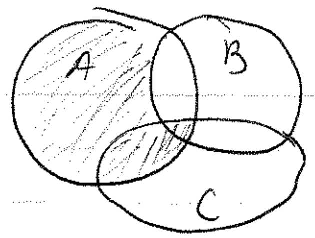
c) If $a \neq 1$ then most choices of b and c work.

For example

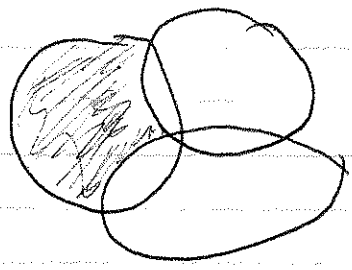
$a=2$, $b=3$, $c=2$

$2^{(3^2)} = 2^9 \qquad (2^3)^2 = 2^6 \qquad 2^9 \neq 2^6.$

d)



$A-(B-C)$



$(A-B)-C$

So the key difference is whether $A \cap C$ is empty or not.

If $A \cap C \neq \phi$ then $A-(B-C) \neq (A-B)-C$.

So let $A=\{1\}$ $B=\{2\}$
$C=\{1\}$

$$A-(B-C) = \{1\}-(\{2\}-\{1\}) = \{1\}-\{2\} = \{1\}$$
$$(A-B)-C = (\{1\}-\{2\})-\{1\} = \{1\}-\{1\} = \phi.$$

Since $\{1\} \neq \phi$ we have our counterexample.

(3) $a \in \mathbb{Z}$, $a \geq 3$, Prove $a^2 > 2a+1$

I'll show 3 proofs:

Proof 1: $a^2 > 2a+1$ if $a^2-2a > 1$

$a^2-2a = a(a-2)$.

Since $a \geq 3$ $a \geq 3$ and $(a-2) \geq 1$

so $a(a-2) \geq (3)(1) = 3$

So $a^2-2a \geq 3 > 1$ so $\boxed{a^2 > 2a+1}$

(Alex + Natalie Solution)

Proof 2: Since $a \geq 3$, then $a = 3+b$ ~~for~~ some $b \geq 0$.

$$a^2 = (3+b)^2 = 9 + 6b + b^2$$
$$2a+1 = 2(b+3)+1 = 2b+7$$

Since $b \geq 0$ $\quad b^2 \geq 0$

$$6b \geq 2b$$
$$9 > 7$$

so $b^2 + 6b + 9 > 2b + 7$

so $\boxed{a^2 > 2a+1}$

Proof 3: Let's prove it by induction

The base case is $a = 3$ so let's check it:

$$a^2 = 3^2 = 9$$
$$2a+1 = 7$$

since $9 > 7$, the base case is correct.

The induction hypothesis would be to assume that

$$k^2 > 2k+1.$$

Now we need to prove $(k+1)^2 > 2(k+1)+1$

so we need to verify $k^2 + 2k + 1 > 2k + 3$

Since $k^2 > 2k+1$ then $k^2 + 2k + 1 > (2k+1) + (2k+1)$
$= 4k+2.$

We know $(k+1)^2 > 4k+2$, we want to show $(k+1)^2 > 2k+3$

One way to prove it would be by showing

$4k+2 > 2k+3$

so $2k > 1$

so $k > 1/2$

But $k \geq 3$ so indeed $(k+1)^2 > 2k+3$ $\boxtimes$

(Another induction solution:

assume $k^2 > 2k+1$ for some $k \geq 3$

now we want $(k+1)^2 > 2k+3$

so $k^2+2k+1 > 2k+3$

so $k^2 > 2$

but $k \geq 3$ so $k^2 \geq 9 > 2$

Therefore $k^2 > 2k+1 \Rightarrow (k+1)^2 > 2k+3$

and the induction follows)

(4) Prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

($\Rightarrow$)

Proof: Let $(x,y) \in A \times (B \cap C)$

so $x \in A$ and $y \in B \cap C$

so $x \in A$ and $y \in B$ and $y \in C$.

so ($x \in A$ and $y \in B$) and ($x \in A$ and $y \in C$)

so $(x,y) \in A \times B$ and $(x,y) \in A \times C$

so $(x,y) \in (A \times B) \cap (A \times C)$.

($\Leftarrow$)

Let $(x,y) \in (A \times B) \cap (A \times C)$

so $(x,y) \in A \times B$ and $(x,y) \in A \times C$

so ($x \in A$ and $y \in B$) and ($x \in A$ and $y \in C$)

so $x \in A$ and $y \in B$ and $y \in C$

so $x \in A \times (B \cap C)$. $\boxtimes$

(5) a) Here's a non-induction proof:

Let $S = 1 + 5 + 9 + \dots (4n-3)$

so $S = (4n-3) + (4n-7) + \dots + 1$

so $2S = (4n-2) + (4n-2) + \dots + (4n-2)$

$= n(4n-2)$

so $S = \dfrac{n(4n-2)}{2} = n(2n-1) = 2n^2 - n.$

Here's an induction proof:

Let $n = 1$, $\underline{1} + 5 + \dots + (4n-3) = 1$

$2n^2 - n = 2(1^2) - 1 = \underline{1}.$

So the base case is true.

Assume it's true for $n = k$, i.e.

$1 + 5 + 9 + \dots + (4k-3) = 2k^2 - k.$

Goal: Prove $1 + 5 + 9 + \dots + (4k-3) + (4(k+1)-3) = 2(k+1)^2 - (k+1)$

so we want

$1 + 5 + 9 + \dots + (4k-3) + (4k+1) = 2(k^2+2k+1) - (k+1)$

$1 + 5 + \dots + (4k-3) = 2k^2 - k$

so

$1 + 5 + \dots + (4k-3) + (4k+1) = 2k^2 - k + 4k + 1 = 2k^2 + 3k + 1$

and $2(k+1)^2 - (k+1) = 2(k^2+2k+1) - (k+1) = 2k^2 + 4k + 2 - k - 1$

$= 2k^2 + 3k + 1.$

So $1 + 5 + \dots + (4k+1) = 2(k+1)^2 - (k+1)$ $\boxed{X}$

b) Prove $1+10+10^2+\dots+10^n = \frac{10^{n+1}-1}{9}$

Pf:

Base case: $n=1$

$$1+10 = 11$$

$$\frac{10^{n+1}-1}{9} = \frac{10^2-1}{9} = \frac{99}{9} = 11.$$ $11=11$ so it works.

Induction Hypothesis:

Assume $1+10+\dots+10^k = \frac{10^{k+1}-1}{9}$.

We want to prove $1+10+\dots+10^{k+1} = \frac{10^{k+2}-1}{9}$

$$\underbrace{1+10+\dots+10^k}+10^{k+1} = \frac{10^{k+1}-1}{9} + 10^{k+1}$$

$$= \frac{10^{k+1}-1+9\cdot 10^{k+1}}{9} = \frac{10\cdot 10^{k+1}-1}{9} = \frac{10^{k+2}-1}{9}.$$

So by induction $1+10+\dots+10^n = \frac{10^{n+1}-1}{9}$.

Note: Here's a non-inductive proof

$1+10+\dots+10^n = \frac{10^{n+1}-1}{9}$ if and only if

$$9(1+10+\dots+10^n) = 10^{n+1}-1.$$

$$\iff$$

$$(10-1)(1+10+\dots+10^n) = 10^{n+1}-1$$

$$(10-1)(1+10+\dots+10^n) = \cancel{10}+\cancel{10^2}+10^3+\dots+\cancel{10^n}+10^{n+1} - (1+\cancel{10}+\cancel{10^2}+\dots+\cancel{10^n})$$

$$= 10^{n+1}-1$$ ✓

6) a) Prove $e^n > n+7$ for $n \geq 3$.

Pf: $e^3 > 10$ so the base case is true.

$(e^3 = 20.0855369232\ldots > 10)$

Assume $e^k > k+7$. (Induction Hypothesis)

Let's prove this implies $e^{k+1} > k+8$

$e^{k+1} = e \cdot e^k$. Since $e^k > k+7$

then $e^{k+1} > e(k+7) = e(k+7)$

Since $e > 2$ $e(k+7) > 2(k+7) = 2k+14$.

Since $2k+14 > k+8$

then $e^{k+1} > k+8$

(because $e^{k+1} > 2k+14$ and $2k+14 > k+8$
so by transitivity $e^{k+1} > k+8$)

$\boxtimes$

b) Prove $n^2 \geq 6n+2$ for $n \geq 7$.

Pf: For $n=7$, $n^2 = 49$, $6n+2 = 44$ so $49 \geq 44$ is true.

Assume that for some $k \geq 7$, $k^2 \geq 6k+2$.
Let's prove

$(k+1)^2 \geq 6(k+1)+2 = 6k+8$.

$(k+1)^2 = k^2 + 2k + 1$

Since $k^2 \geq 6k+2$

then

$$
\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&\geq (6k+2) + 2k + 1 \\
&= 8k+3.
\end{aligned}
$$

we want to prove $(k+1)^2 \geq 6k+8$

so we may verify

$$8k+3 \geq 6k+8$$

$\cancel{\Leftrightarrow}\!=\!7$ $\quad 2k \geq 5 \quad \Leftrightarrow \quad k \geq 5/2$

Since $k \geq 7$, $8k+3 \geq 6k+8$

Since $(k+1)^2 \geq 8k+3$ and $(8k+3) \geq 6k+8$

then

$$(k+1)^2 \geq 6k+8$$

which is what we wanted to prove $\boxtimes$

(7) Prove $\sqrt{2}$ is irrational.

Proof: For the sake of contradiction,

suppose $\sqrt{2}$ is rational, i.e.

there exist $m, n \in \mathbb{Z}$ s.t $\sqrt{2} = \frac{m}{n}$.

Any fraction can be reduced to lowest terms, so if $\sqrt{2} = \frac{m}{n}$ then $\sqrt{2} = \frac{p}{q}$ where p and q don't share any common factors greater than one

(for example $\frac{42}{28} = \frac{21}{14} = \frac{3}{2}$ and 2 and 3 share no common factors)

Now since $\sqrt{2} = p/q$ then

$$2 = \frac{p^2}{q^2} \quad \text{so} \quad p^2 = 2q^2.$$

Since $q \in \mathbb{Z} \Rightarrow q^2 \in \mathbb{Z}$ so p^2 is even.

odd x odd = odd so if p^2 is even, p must be even.

So ~~p^2~~ p is even, i.e. $\exists k \in \mathbb{Z}$ s.t $p = 2k$.

Since $p^2 = 2q^2 \Rightarrow (2k)^2 = 2q^2$

so $4k^2 = 2q^2$ so $2k^2 = q^2$ so q^2 is even.

So $\underline{q \text{ is even}}$.

Since p and q are even they share 2 as a common factor.

This contradicts that $\frac{p}{q}$ was in lowest terms.

So $\sqrt{2}$ is not rational.

So $\sqrt{2}$ is irrational $\boxtimes$

(8) a) $R = \{(1,1), (2,2), (3,3)\}$

Reflexive, symmetric, antisymmetric and transitive.

b) $R = \{(1,1), (2,2), (3,3), (1,2)\}$

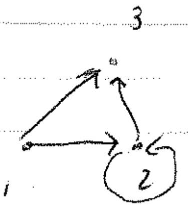
Reflexive, antisymmetric, transitive

c) $R = \{(1,1), (2,2), (1,2), (2,1)\}$

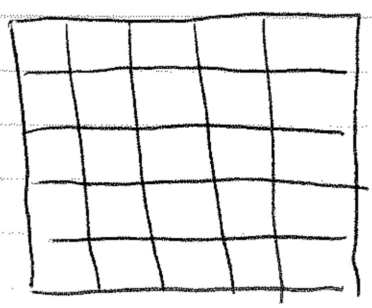
~~Refl~~ Symmetric, Transitive.

d) $R = \{(1,2), (1,3), (2,3), (2,2)\}$

Antisymmetric, transitive.

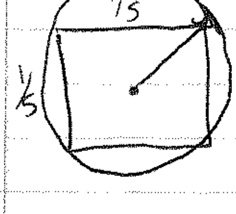


(9)



Divide the 1×1 board in 25 equal squares of $1/5 \times 1/5$.

By the Pigeonhole Principle at least 3 insects will be in one of those 25 squares.

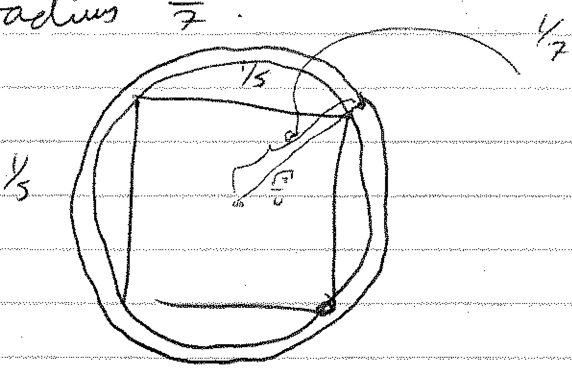


So those 3 insects are inside a circle of radius $\frac{\sqrt{2}}{10}$.

Note that $\frac{\sqrt{2}}{10}$ is half the diagonal of the ~~8~~ $\frac{1}{5} \times \frac{1}{5}$ square.

$\frac{\sqrt{2}}{10} < \frac{1}{7}$ because $98 < 100$ $\left(98 = (7\sqrt{2})^2 \text{ and } 100 = 10^2 \right)$

so these 3 insects are also inside a circle of radius $\frac{1}{7}$.



(10) a) True by definition

b) False. We proved in class that $|\mathbb{N}| < |\mathbb{R}|$.

c) True. We proved in class that $|2^{\mathbb{N}}| = |(0,1)|$

and $|(0,1)| = |\mathbb{R}|$

so $|2^{\mathbb{N}}| = |\mathbb{R}|$.

d) True. $[0,1]$ is $(0,1)$ plus the points 0 and 1.

So it's only slightly "bigger" than $(0,1)$.

we know $|(0,1)| = |\mathbb{R}|$ and $[0,1] \subseteq \mathbb{R}$

so $|[0,1]| \leq |\mathbb{R}|$, while $|[0,1]| \geq |(0,1)| = |\mathbb{R}|$

so $|[0,1]| = |\mathbb{R}| = |(0,1)|$.

e) False. $f(x)=2x-1$ is a bijection from $(0,1)$ to $(-1,1)$

or from $(\frac{1}{2}, \frac{3}{2})$ to $(0,2)$

but never from $(0,1)$ to $(0,2)$.

f) True, it was a homework exercise.

g) False, there is no one-to-one $f: A \to B$ or

no onto $f: B \to A$

but there are one-to-one $f: B \to A$

and onto $f: A \to B$.

h) True, theorem 25.4 in the book (paraphrased)

i) False. If A and B are finite, then it is true, but for infinite sets it's not necessarily true.

Example:

Let $A = B = \mathbb{Z}$

$f: A \to B$, $f(x) = 2x$

and $g: B \to A$, $g(x) = 2x$

Both f and g are 1-1 but neither is onto.

j) False. It's a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$.

Assume that $x, y \in \mathbb{Z}$ for all parts of 11.

(11) a) $f = \{(x,y) \mid x+y=0\}$

is a function because for every x there is a unique, namely $y=-x$.

b) $f = \{(x,y) \mid xy=0\}$

is not a function because if $x=0$ then y can be anything, so in that case y is not defined uniquely.

c) $f = \{(x,y) \mid x \mid y\}$

is not a function because $1 \mid 2$ and $1 \mid 3$ so for $x=1$ there are many choices of y, so again it's not defined uniquely.

(12) a) $f = \{(x,y) \mid x+y=0\}$.

so $y=-x$ so $f(x)=-x$.

Let's prove f is 1-1. Suppose $f(x_1)=f(x_2)$

then $-x_1=-x_2$ so $x_1=x_2$

so f is 1-1 ☒

b) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=7x-12$.

Suppose $f(x)=f(y)$ so $7x-12=7y-12$

so $7x=7y$ so $x=y$.

so f is 1-1.

c) $f = \{(1,1), (2,3), (3,2), (4,3)\}$

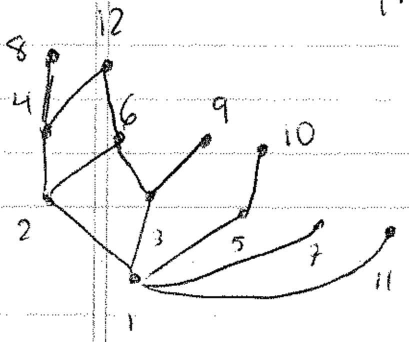
$f(2) = f(4)$ yet $2 \neq 4$ so f is not 1-1.

13) P is the poset on $\{1, 2, \ldots, 12\}$ with the "divides" relation

a) P does not have a maximum because it has incomparable elements with any candidate to be a maximum.

It does have a minimum, namely 1.
1 is below every other number.

b) 7, 8, 9, 10, 11, 12 are all maximal elements because they have no numbers above them.



c) The only minimal is 1.

d) The height is 4.

e) The width is 5.

f) No because it has incomparable elements.

(14) To prove that it is an equivalence relation we need to prove it's reflexive, symmetric and transitive.

Reflexive: Let $P = (X, \preceq)$ be a finite poset then

we want to show $P \cong P$ isomorphic.

Let $f : X \to X$ be $f(x) = x$.

Clearly f is a bijection (since $f(x) = f(y)$ means $x = y$ so f is 1-1

and $\forall\, y \in X$ $f(y) = y$ so f is onto).

To have f be the isomorphism, we'd need to check that f is order preserving i.e.

$$\forall\, a, b \in X \quad a \preceq b \iff f(a) \preceq' f(b)$$

but $f(a) = a$ and $f(b) = b$ so $a \preceq b \iff f(a) \preceq' f(b)$ and $\preceq' = \preceq$

translates to $a \preceq b \iff a \preceq b$ which is clearly true.

So the relation is reflexive.

Symmetric: Let $P = (X, \leq)$ and $Q = (Y, \leq')$

suppose $P \cong Q$ so $\exists$ bijection $f : X \to Y$

such that $\forall a, b \in X$ $a \leq b \iff f(a) \leq' f(b)$.

Let's show $Q \cong P$.

Let g be the inverse of f.

So $g : Y \to X$ and $f \circ g = id_Y$ and $g \circ f = id_X$

g is a bijection by construction (it's the inverse of a bijection).

We need only check g is order preserving.

For the sake of contradiction suppose $a, b \in Y$

and ~~$g(a) \leq g$~~ $a \leq' b$ yet $g(a) \not\leq g(b)$

$g(a) \in X$ and $g(b) \in X$ and since $g(a) \not\leq g(b)$

then $g(a) > g(b)$ so $f(g(a)) >' f(g(b))$

since f is order-preserving

but $f(g(a)) = a$ and $f(g(b)) = b$

so $a >' b$ but $a \leq' b$ $\Rightarrow\Leftarrow$

Therefore g is order-preserving so

g is a poset isomorphism so

$Q \cong P$ so $\cong$ is symmetric

Now we're done to proving transitivity:

Suppose $P = (X, \leq)$, $Q = (Y, \leq')$, $T = (Z, \leq'')$ are finite posets s.t $P \cong Q$ and $Q \cong T$.

Goal: Prove $P \cong T$.

Since $P \cong Q$ $\exists$ $f : X \to Y$, a bijection that is order-preserving.

Since $Q \cong T$ $\exists$ $g : \cancel{X} \to Z$, a bijection that is order preserving.

Since f and g are bijections then

$$g \circ f : X \to Z \quad \text{is a bijection.}$$

We do need to verify that $g \circ f$ is order-preserving.

~~For the sake of contradiction, assume it is not,~~

~~so $\exists$ $a, b \in X$ such that $a \leq b$ and $g \circ f(a) >'' g \circ f(b)$~~

We need to prove $a \leq b \iff g \circ f(a) \leq'' g \circ f(b)$.

($\Rightarrow$) Suppose $a \leq b$, then since f is order-preserving

$f(a) \leq' f(b)$, now since g is order-preserving

$g(f(a)) \leq'' g(f(b))$ so $g \circ f(a) \leq'' g \circ f(b)$.

($\Leftarrow$) Suppose $g \circ f(a) \leq'' g \circ f(b)$ so $g(f(a)) \leq'' g(f(b))$

Since g is order preserving $f(a) \leq' f(b)$ and since f is order-preserving $a \le b$ $\boxed{A}$

This shows $g \circ f$ is order-preserving, so $P \cong T$
so $\cong$ is transitive so $\cong$ is an equivalence relation $\boxed{A}$

(15) To prove $(X \times X, \le)$ is a linear order
we need to show every two elements of $X \times X$
are comparable.

Suppose (x_1, y_1) and (x_2, y_2) are two elements
of $X \times X$.
We need to show they are comparable.

Since X is a total order $x_1 < x_2$, $x_1 = x_2$ or $x_1 > x_2$.
If $x_1 < x_2$ then $(x_1, y_1) \le (x_2, y_2)$ by definition, so
in that case they are comparable.
If $x_1 > x_2$ then $(x_2, y_2) \le (x_1, y_1)$ by the definition
of $\le$ in the exercise so again (x_1, y_1) and (x_2, y_2)
are comparable.
We're left with $x_1 = x_2$.
Since X is a linear order then $y_1 \le y_2$ or $y_2 \ge y_1$
If $y_1 \le y_2$ then by definition $(x_1, y_1) \le (x_2, y_2)$
If $y_2 \le y_1$ then by definition $(x_2, y_2) \le (x_1, y_1)$.

So in all cases (x_1, y_1) and (x_2, y_2) are
comparable.

So $(X \times X, \le)$ is a linear order $\boxed{A}$