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MATH 230 MIDTERM #1
September 30, 2013

INSTRUCTIONS: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

• You may NOT use a calculator.
• Show all of your work.

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Official Cheat Sheet

1. Let $A$ be a set. Then $2^A$ is the set of all subsets of $A$. For example, if $A = \{1, 2\}$, then $2^A = \emptyset, \{1\}, \{2\}, \{1, 2\}$.

2. $|A|$ is the number of elements of $A$. A useful formula is: $|A \cup B| = |A| + |B| - |A \cap B|$ if $A$ and $B$ are finite sets. Another useful formula is $|2^A| = 2^{|A|}$ when $A$ is finite.

3. Here are some Boolean algebra properties (which can be translated easily to set properties by equating $\lor$ with $\cup$ and $\land$ with $\cap$):
   
   - $x \land y = y \land x$ and $x \lor y = y \lor x$.
   - $(x \land y) \land z = x \land (y \land z)$ and $(x \lor y) \lor z = x \lor (y \lor z)$.
   - $x \land (y \lor z) = (x \land y) \lor (x \land z)$ and $x \lor (y \land z) = (x \lor y) \land (x \lor z)$.

4. $\mathbb{Z}$ is the set of integers. $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of positive integers.

5. Let $A$ and $B$ be sets. Then
   
   - $A \cup B = \{x | x \in A \text{ or } x \in B\}$,
   - $A \cap B = \{x | x \in A \text{ and } x \in B\}$,
   - $A - B = \{x | x \in A \text{ and } x \not\in B\}$,
   - $A \Delta B = (A - B) \cup (B - A)$.
   - $A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$.

6. Let $a$ and $b$ be integers.
   
   - $a$ is even if there exists an integer $c$ such that $a = 2c$.
   - $a$ is odd if there exists an integer $c$ such that $a = 2c + 1$.
   - We say $a|b$ ($a$ divides $b$) if there exists an integer $c$ such that $b = ac$.
   - $a$ is composite if $|a| > 1$ and there exists $c$ such that $1 < c < |a|$ and $c|a$.
   - $a$ is prime if $a > 1$ and $a$ is not composite.
   - $a$ is perfect if $a$ equals the sum of its positive divisors less than $a$. 
1. True or False (Just answer true or false, you don’t need to explain your answer):

(a) [2 points] -23 is prime.  
**FALSE**  
(to be prime it would have to be positive)

(b) [2 points] 7|1001.  
**TRUE**  
\( \frac{1001}{7} = 143 \)

(c) [2 points] The sum of two odd numbers is odd.  
**FALSE**  
(The sum is even)

(d) [2 points] \( T \subseteq A \) if and only if \( T \in 2^A \).  
**TRUE**  
(by definition)

(e) [2 points] \( \emptyset \subseteq \{\emptyset\} \).  
**TRUE**  
(the empty set is the subset of anything)

(f) [2 points] Let \( n = 2^{p-1}(2^p - 1) \) where \( 2^p - 1 \) is prime. \( n \) is a perfect number.  
**TRUE**  
(Proved in class)

(g) [2 points] \( 2 \in \{1, 2, \{1, 2\}\} \).  
**TRUE**

(h) [2 points] If \( x^2 < 0 \), then \( x \) is a perfect number.  
**FALSE**  
(No \( x \) satisfies \( x^2 < 0 \), so it is a vacuous statement)

(i) [2 points] Two right triangles that have hypotenuses of the same length have the same area.  
**FALSE**  
(Example: \( \sqrt{5} \) has area 6, \( \sqrt{2} \) has area 3, so \( \sqrt{\frac{5}{2}} \) has area \( \sqrt{\frac{5}{2}} \))

(j) [2 points] \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = 1 \).  
**FALSE**  
(For \( x = 2 \), there is no \( y \in \mathbb{Z} \) because \( \frac{1}{2} \) is not an integer)
2. For the following pairs of statements $A$, $B$, write $a$ if the statement “If $A$, then $B$” is true, write $b$ if the statement “If $B$, then $A$” is true and write $c$ if the statement ”$A$ if and only if $B$ is true”. You should write all that apply.

(a) [5 points] $A$: $x > 0$. $B$: $x^2 > 0$.

$$a$$

(b) [5 points] $A$: Ellen is a grandmother. $B$: Ellen is female.

$$a$$

(c) [5 points] $A$: $x$ is odd. $B$: $x + 1$ is even.

$$a, b, c$$

(d) [5 points] $A$: Polygon $PQRS$ is a rectangle. $B$: Polygon $PQRS$ is a square.

$$b$$
3. Proofs:

(a) [5 points] Using the definition of odd integer provided in the “cheat sheet”, prove that if \( n \) is an odd integer, then \(-n\) is also an odd integer.

**Proof:** Let \( n \) be an odd integer. Therefore there exists an integer \( c \) such that \( n = 2c + 1 \).

\[-n = -2c - 1 = -2c - 2 + 1 = 2(-c - 1) + 1 = 2d + 1\]

where \( d = -c - 1 \).

Therefore there exists an integer \( d \) such that \(-n = 2d + 1\).

Therefore \(-n\) is an odd integer. \( \square \)

(b) [5 points] Let \( a, b \) and \( d \) be integers. Suppose \( b = aq + r \) where \( q \) and \( r \) are integers. Prove that if \( d \mid a \) and \( d \mid b \), then \( d \mid r \).

**Proof:** Let \( a, b, d, q, r \) be integers such that \( b = aq + r \) and \( d \mid a \) and \( d \mid b \).

Since \( d \mid a \) there exists an integer \( m \) such that \( a = dm \).

Since \( d \mid b \) there exists an integer \( n \) such that \( b = dn \).

Therefore

\[ r = b - aq = dn - (dm)q = d(n - mq) = dc \]

for \( c = n - mq \).

Therefore there exists an integer \( c \) such that \( r = dc \).

Therefore \( d \mid r \). \( \square \)
4. Find counterexamples to disprove the following statements:

(a) [5 points] An integer $x$ is positive if and only if $x + 1$ is positive.

$x = 0$ is a counterexample since $x + 1 = 1$ is positive, yet 0 is not positive.

(b) [5 points] An integer is a palindrome if it reads the same forwards and backwards when expressed in base 10. For example, 1331 is a palindrome. All palindromes are divisible by 11.

101 is a palindrome, yet it is not divisible by 11, so it is a counterexample.

(c) [5 points] If $a$, $b$ and $c$ are positive integers then $a^{(bc)} = (a^b)^c$.

Let $a = 2$, $b = 3$, $c = 2$

$2^{(3^2)} = 2^9$  \hspace{1cm} (2^3)^2 = 2^6 \hspace{1cm} 2^9 \neq 2^6$

so $a = 2$, $b = 3$, $c = 2$ is a counterexample.

(d) [5 points] Let $A$, $B$ and $C$ be sets. Then $A - (B - C) = (A - B) - C$.

Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{2, 3\}$.

$A - (B - C) = \{1, 2\} - (\{2, 3\} - \{2, 3\}) = \{1, 2\} - \emptyset = \{1, 2\}$

$(A - B) - C = (\{1, 2\} - \{2, 3\}) - \{2, 3\} = \{1, 2\} - \{2, 3\} = \{1\}$

As easily seen, this is a counterexample since $A - (B - C) \neq (A - B) - C$ in this example.
5. Boolean Algebra

(a) [5 points] Prove or disprove that the Boolean expressions \( x \rightarrow \neg y \) and \( \neg(x \rightarrow y) \) are logically equivalent.

\[
\begin{array}{c|c|c|c|c|c}
    x & y & \neg y & x \rightarrow \neg y & x \rightarrow y & \neg(x \rightarrow y) \\
    \hline
    T & T & F & F & T & F \\
    T & F & T & T & F & T \\
    F & T & F & T & T & F \\
    F & F & T & T & T & F \\
\end{array}
\]

They are not logically equivalent since \( x = F \) fails \( x \rightarrow \neg y = T \) while \( \neg(x \rightarrow y) = F \).

(b) [5 points] The expression \( x \rightarrow y \) can be rewritten in terms of only the basic operations \( \land, \lor \) and \( \neg \); that is, \( x \rightarrow y = (\neg x) \lor y \). Find an expression that is logically equivalent to \( x \leftrightarrow y \) that uses only the operations \( \land, \lor, \neg \) and prove that your expression is correct.

\[
x \leftrightarrow y = \left( (\neg x) \lor y \right) \land \left( x \lor (\neg y) \right)
\]

\[
\begin{array}{c|c|c|c|c|c}
    x & y & x \leftrightarrow y & (\neg x) \lor y & x \lor (\neg y) & (\neg x) \lor y \land x \lor (\neg y) \\
    \hline
    T & T & T & T & T & T \land T = T \\
    T & F & F & F & F & F \land F = F \\
    F & T & F & F & T & T \land F = F \\
    F & F & T & T & T & T \land T = T \\
\end{array}
\]

Indeed, they are logically equivalent.

Another possibility that works is \( (x \land y) \lor (\neg x) \land (\neg y) \).
6. Consider the following proposition. Let $N$ be a two-digit number and let $M$ be the number formed from $N$ by reversing the digits of $N$. Now compare $N^2$ and $M^2$. The digits of $M^2$ are precisely those of $N^2$, but reversed. For example:

\[
\begin{align*}
10^2 &= 100 & 01^2 &= 001 \\
11^2 &= 121 & 11^2 &= 121 \\
12^2 &= 144 & 21^2 &= 441 \\
13^2 &= 169 & 31^2 &= 961
\end{align*}
\]

and so on. Here is a proof of the proposition:

**Proof.** Since $N$ is a two-digit number, we can write $N = 10a + b$ where $a$ and $b$ are the digits of $N$. Since $M$ is formed from $N$ by reversing digits, $M = 10b + a$.

Note that $N^2 = (10a + b)^2 = 100a^2 + 20ab + b^2 = (a^2) \times 100 + (2ab) \times 10 + (b^2) \times 1$, so the digits of $N^2$ are, in order, $a^2, 2ab, b^2$.

Likewise, $M^2 = (10b + a)^2 = (b^2) \times 100 + (2ab) \times 10 + (a^2) \times 1$, so the digits of $M^2$ are, in order, $b^2, 2ab, a^2$, exactly the reverse of $N^2$, which completes the proof.

(a) [5 points] Prove that the proposition is false.

\[
\begin{align*}
15^2 &= 225 \\
51^2 &= 2500 + 100 + 1 = 2601
\end{align*}
\]

225 and 2601 are not just reversing the digits.

(b) [5 points] Explain why the proof is invalid.

$a^2, 2ab, b^2$ are digits if they are number less than or equal to 9.

If $a$ or $b$ is at least 4 then $a^2, b^2$ are greater than 9, so they are not digits. The proof doesn't take this into account.
7. Counting

(a) [5 points] In how many ways can we make a list of three integers \((a, b, c)\) where \(0 \leq a, b, c \leq 9\) such that \(a + b + c\) is even?

\[
\begin{align*}
\text{a has 10 choices} & \quad \text{(any number)} \\
\text{b has 10 choices} & \quad \text{(any number)} \\
\text{c has 5 choices} & \quad \text{(an even number if \(a+b\) is even,)} \\
& \quad \text{an odd number if \(a+b\) is odd.)}
\end{align*}
\]

\[\boxed{10 \times 10 \times 5 = 500.}\]

(b) [5 points] In how many ways can we make a list of three integers \((a, b, c)\) where \(0 \leq a, b, c \leq 9\) such that \(abc\) is even?

\[\boxed{10^3 - 5^3}\]

because for \(abc\) not to be even \(a, b, c\) must be odd. There are \(10^3\) lists \((a, b, c)\) and \(5^3\) of them have \(a, b, c\) all odd.
8. Evaluate the following:

(a) [5 points] \[ \prod_{k=1}^{100} \frac{k+1}{k} \]

\[
\frac{2 \cdot 3 \cdot \ldots \cdot 101}{1 \cdot 2 \cdot \ldots \cdot 100} = \quad \boxed{101}
\]

(b) [5 points] \[ \prod_{k=0}^{100} \frac{k^2}{k+1} \]

\[
\frac{0^2 \cdot 1^2 \cdot 2^2 \cdot \ldots \cdot 100^2}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 100 \cdot 101} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 100}{101} = \boxed{\frac{100!}{101}}
\]
9. Let \( A \times B = \{(1, 2), (1, 3), (1, 7), (2, 2), (2, 3), (2, 7)\} \).

(a) [5 points] What is \( A \cup B \)?

\[
A = \{1, 2\} \\
B = \{2, 3, 7\}
\]

\[ A \cup B = \{1, 2, 3, 7\} \]

(b) [5 points] What is \( A \cap B \)?

\[ A \cap B = \{2\} \]

(c) [5 points] What is \( A - B \)?

\[ A - B = \{1\} \]

(d) [5 points] What is \( A \Delta B \)?

\[
(A - B) \cup (B - A) \\
= \{1\} \cup \{3, 7\} = \{1, 3, 7\}
\]
10. [10 points] Let \( A, B \) and \( C \) be sets. Prove that
\[
(A \cup B) - C = (A - C) \cup (B - C).
\]

**Proof:**

\[ (\Rightarrow) \]
\[ \text{Let } x \in (A \cup B) - C \]
\[ \text{then } x \in A \cup B \text{ and } x \notin C \]
\[ \text{then } x \in A \text{ or } x \in B \text{ and } x \notin C \]

Therefore \( (x \in A \text{ and } x \notin C) \text{ or } (x \in B \text{ and } x \notin C) \)

Therefore \( x \in (A - C) \cup (B - C) \).

\[ (\Leftarrow) \]
\[ \text{Let } x \in (A - C) \cup (B - C) \]
\[ \text{Then } x \in (A - C) \text{ or } x \in (B - C) \]

\[ \text{so } (x \in A \text{ and } x \notin C) \text{ or } (x \in B \text{ and } x \notin C) \]

\[ \text{so } (x \in A \text{ or } x \in B) \text{ and } x \notin C \]

\[ \text{so } x \in (A \cup B) - C \]

**Alternative proof using Boolean algebra.**

Let \( m = x \in A \), \( n = x \in B \), \( r = x \notin C \).

In general
\[ (m \lor r) \lor (n \land r) = (m \lor n) \land r \quad \text{(Distributive Property)} \]

So
\[ ((x \in A) \land (x \notin C)) \lor ((x \in B) \land (x \notin C)) = ((x \in A) \lor (x \in B)) \land (x \notin C) \]

Therefore
\[ (A - C) \cup (B - C) = (A \cup B) - C \]