Midterm 2

SOLUTIONS

1. a) T $\quad 10 \mid 3-23=-20$

b) F $\quad 7 \nmid 95-111=-16$

c) T $\quad a<b \wedge b<c \Rightarrow a<c$.

d) F $\quad R=\{(1,1),(2,2),(3,3)\}$ on $\{1,2,3\}$ is both

e) T $\quad$ If $R \neq \emptyset$ then you can't be both.

f) F $\quad$ Problem 5a is an example of a reflexive and symmetric relation that is not transitive.

g) T $\quad$ (Proved in class)

h) T

i) T $\quad$ (It's the base case)

j) F $\quad$ (The difference is in the induction hypothesis)

2. a) Base case: $n=1$

$$1+2=3$$

$$2^{1+1}-1=3$$

So it's true.

Suppose $1+2+2^2+\dots+2^k=2^{k+1}-1$

Goal: $1+2+2^2+\dots+2^k+2^{k+1}=2^{k+2}-1$

$$1+2+2^2+\dots+2^k+2^{k+1}=2^{k+1}-1+2^{k+1}$$

$$=2\cdot 2^{k+1}-1$$

$$=2^{k+2}-1.$$

Hence it's true by induction.

b) For $n=1$ we have $3 = 2(1)^2 + 1 = 3$

Suppose $3 + 7 + 11 + \ldots + (4k-1) = 2k^2 + k$

then $3 + 7 + 11 + \ldots + (4k-1) + (4k+3) = 2k^2 + k + 4k + 3$

$= 2k^2 + 5k + 3$

$= 2k^2 + 4k + 2 + (k+1) = 2(k+1)^2 + (k+1)$ $\boxtimes$

c) Base case: $n=3$: $9 = 3^2 > 2(3) + 1 = 7$

Suppose $k^2 > 2k+1$

then $(k+1)^2 = k^2 + 2k + 1 > (2k+1) + 2k + 1 = 4k + 2$

Since $k \geq 3$ $2k \geq 6$

so $4k + 2 = 2k + 2k + 2 \geq 2k + 8 > 2(k+1) + 1$

so $(k+1)^2 > 2(k+1) + 1$ so the statement is true for all $n \geq 3$. $\boxtimes$

d) $2^5 = 32 > 25 = 5^2$ so it's true for $n=5$.

Suppose $2^k > k^2$.

Then $2^{k+1} = 2 \cdot 2^k > 2k^2 = k^2 + k^2$

We want to show $2k^2 > (k+1)^2 = k^2 + 2k + 1$

so we want to show $k^2 > 2k+1$,

but from (c) we know $k^2 > 2k+1$ for $k \geq 3$.

Since $k \geq 5$, $k^2 > 2k+1$ so $2k^2 > (k+1)^2$

so $2^{k+1} > (k+1)^2$

$\boxtimes$

(3) a) Suppose there exist 4 consecutive integers whose sum is divisible by 4.

b) Suppose there is a point in P that is not in a common line with the rest.

(4) a) For the sake of contradiction, suppose there exist 4 consecutive integers whose sum is not divisible by 4.
Suppose the 4 consecutive integers are $n, n+1, n+2, n+3$
So $4 \mid n+(n+1)+(n+2)+(n+3) = 4n+6$

$$4n+6 = 4(n+1)+2$$

Since $4 \mid 4n+6$, $\exists\, k$ s.t $4n+6 = 4k$.

So $6 = 4(k-n)$

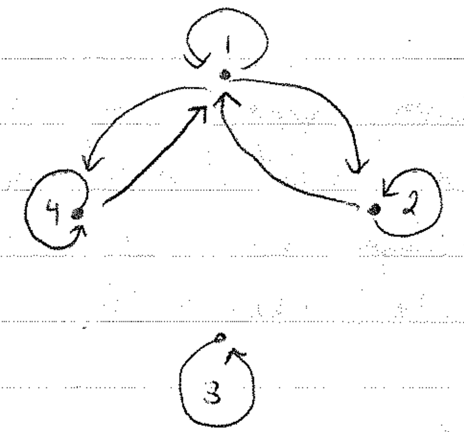
so $\frac{3}{2} = k-n \in \mathbb{Z}$

But $k-n \notin \mathbb{Z}$. Therefore our assumption must be false, hence there do not exist 4 consecutive integers whose sum is divisible by 4.

b) Suppose not all points are lying in the same line.
Take 2 points. If all points lied on the same line, the line would include those 2 points. Take a point not on that line. Since not all points lie in the same line there is a point not on the same line as those 2 points.

But then those 2 points together with the point not on the line between those two points are 3 points that are not colinear. This contradicts our assumption that ~~all~~ any 3 points are colinear. $\boxtimes$

(5) a)



$\boxed{\text{Reflexive, Symmetric}}$

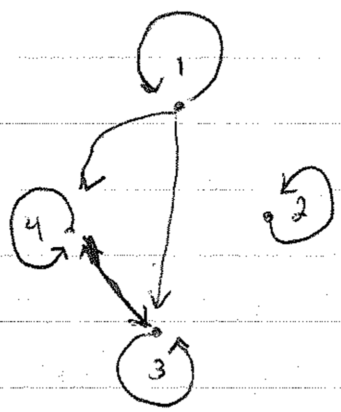
Not transitive

$(2,1) \wedge (1,4) \rightarrow (2,4)$ but $(2,4) \notin R$

Not irreflexive since $(1,1) \in R$

Not anti-symmetric since $(1,2) \in R \wedge (2,1) \in R$

b)



$\boxed{\text{Reflexive, Transitive}}$ $\boxed{\text{Antisymmetric}}$

Not irreflexive

Not symmetric

(6) a) $[4] = \{1, 2, 4\}$

$[1] = \{1, 2, 4\}$

b) $[\{1,2,3\}] = \{ \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{3,4,5\} \}.$

c)

c) $[-3] = \{-3, 3\}$.

$[0] = \{0\}$.

d) $[(0,1)] = \{(a,b) : a, b \in \mathbb{R},\ a^2+b^2 = 0^2+1^2 = 1\}$

$= \{(x,y) : x, y \in \mathbb{R},\ x^2+y^2 = 1\}$

It's the unit circle. (Circle of radius 1 centered at the origin).

7) Let's start with reflexive.

$(a,b) R (a,b)$ because $a^2+b^2 = a^2+b^2$

Hence R is reflexive.

SYMMETRIC:

Suppose $(a,b) R (c,d)$ then $a^2+b^2 = c^2+d^2$

so $c^2+d^2 = a^2+b^2$

so $(c,d) R (a,b)$, so R is symmetric

TRANSITIVE

Suppose $(a,b) R (c,d)$ and $(c,d) R (e,f)$

Goal: Prove $(a,b) R (e,f)$.

$(a,b) R (c,d) \Rightarrow a^2+b^2 = c^2+d^2$

$(c,d) R (e,f) \Rightarrow c^2+d^2 = e^2+f^2$

$\Rightarrow a^2+b^2 = e^2+f^2$

$\Rightarrow (a,b) R (c,d) \Rightarrow R$ is transitive

$\Rightarrow R$ is an equivalence relation.