Poker Hands

July 24, 2013

A deck of cards has 52 cards, 13 numbers (A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K), 4 suits (spades, clubs, hearts, diamonds). A poker hand consists of 5 cards from the deck. A pair means two cards have the same number. Three of a kind, means three of the same number, similarly defined for four of a kind. A Full House is when the poker hand consists of a three-of-a-kind and a pair (example AA222). A straight means you have five consecutive numbers, the first possible straight is A - 2 - 3 - 4 - 5, the last one is 10 - J - Q - K - A. A flush means the five cards are of the same suit. A straight flush means the hand is both a flush and a straight.

1. How many poker hands are there? (For example, one poker hand is A of hearts, 2 of spades, J of hearts, 5 of clubs and A of diamonds.

\[
\binom{52}{5} = 2,598,960
\]
2. How many poker hands are there with just one pair?

\[
\frac{52 \times 3 \choose 2}{2} \times \frac{48 \times 3 \times 4 \times 4 \times 4 \times 0}{3!} = 1098240
\]

or

\[
\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1}^3
\]

pick the pair
pick the suits for the pair
pick the other three numbers
pick their suits.

3. How many poker hands with two pairs are there?

\[
\frac{52 \times 3 \choose 2}{2} \times \frac{48 \times 3 \choose 2} \times 4 \times 4 = 123552
\]

or

\[
\binom{13}{2} \binom{4}{2}^2 \binom{11}{3} \binom{4}{1}
\]
4. How many poker hands with three of a kind are there (and not a full house)?

\[
\frac{\binom{52}{3} \times 3!^2}{3!} \times \frac{4 \times 4^4}{2!} = 54,912
\]

Or

\[
\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1}^2
\]

5. How many poker hands with four of a kind are there?

\[
\frac{52 \times 3 \times 2 \times 1}{4!} \times 4^8 = 624,974
\]

Or

\[
\binom{13}{1} \binom{4}{4} \binom{12}{1} \binom{4}{1}
\]
6. How many poker hands are a full house?

\[
\frac{52 \times 3 \times 2}{3!} \times \frac{48 \times 3}{2!} = 3744
\]

or

\[
\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2}
\]

7. How many poker hands are a straight flush?

\[
10 \binom{4}{1} = 40
\]

\[
\begin{align*}
A & \ 2 \ 3 \ 4 \ 5 \\
2 & \ 3 & \ 4 & \ 5 & \ 6 \\
\vdots & & & & \\
T & \ J & \ Q & \ K & \ A
\end{align*}
\]

4 suits for each
8. How many poker hands are a straight but not a flush?

\[ 10 \left( \binom{4^5}{1} \right) - 40 = 10200. \]

The straight sets the suits for the straights.

9. How many poker hands are a flush but not a straight?

\[ 4 \left( \binom{13}{5} \right) - 40 = \frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!} - 40 = 5708 \]
10. How many poker hands are left (no pair, no flush, no straight)?

\[
\binom{52}{5} - \text{pairs} - \text{two pair} - \text{three of a kind} - \text{straight} - \text{flush} - \text{straight flush}
\]

\[= 1302540.\]

11. Can you give a hierarchy to the poker hands from less common to most common?

<table>
<thead>
<tr>
<th>Hand</th>
<th>#</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight Flush</td>
<td>40</td>
<td>0.00154%</td>
</tr>
<tr>
<td>Four of a Kind</td>
<td>624</td>
<td>0.024%</td>
</tr>
<tr>
<td>Full House</td>
<td>3744</td>
<td>0.144%</td>
</tr>
<tr>
<td>Flush</td>
<td>5108</td>
<td>0.1965%</td>
</tr>
<tr>
<td>Straight</td>
<td>10200</td>
<td>0.392465%</td>
</tr>
<tr>
<td>Three of a Kind</td>
<td>54912</td>
<td>2.113%</td>
</tr>
<tr>
<td>Two Pair</td>
<td>123552</td>
<td>4.754%</td>
</tr>
<tr>
<td>Pair</td>
<td>1098240</td>
<td>42.257%</td>
</tr>
<tr>
<td>Nothing</td>
<td>1302540</td>
<td>50.12%</td>
</tr>
</tbody>
</table>