Practice Exam 2
Solutions

(1) a)

Base case: LHS: $4(1)-3 = 1$

RHS: $2(1)^2 - 1 = 1$

so LHS = RHS so it works.

Induction Hypothesis: Suppose $1+5+9+\dots+(4k-3) = 2k^2-k$.

$$1+5+9+\dots+(4k-3)+(4k+1) = 2k^2-k+4k+1$$
$$= 2k^2+3k+1$$
$$= 2k^2+4k+2-(k+1)$$
$$= 2(k^2+2k+1)-(k+1)$$
$$= 2(k+1)^2-(k+1)$$

so by induction the theorem is proved.

b) Base case: $n=1$: $1+10=11$

$$\frac{10^{1+1}-1}{9} = \frac{10^2-1}{9} = \frac{99}{9} = 11$$

So the base case is true.

Suppose $1+10+10^2+\dots+10^k = \dfrac{10^{k+1}-1}{9}$.

Then $1+10+10^2+\dots+10^k+10^{k+1} = \dfrac{10^{k+1}-1}{9} + 10^{k+1}$

$$= \frac{10^{k+1}-1+9\cdot 10^{k+1}}{9} = \frac{10\cdot 10^{k+1}-1}{9} = \frac{10^{k+2}-1}{9}$$

so by induction $1+10+\dots+10^n = \dfrac{10^{n+1}-1}{9}$.

(2) a) Base case: $n=3$, $e^3 > (2.5)^3 > 2(2.5)^2 = 2(6.25) = 12.5$

$n+7 = 3+7 = 10$

$e^3 > 12.5 > 10 = (3+7)$

so it's true for $n=3$.

Suppose $e^k > k+7$ for some $k \geq 3$.

Let's show $e^{k+1} > (k+1)+7 = k+8$

$$e^{k+1} = e(e^k) > e(k+7) = ek + 7e$$
$$> 2k+14 \quad 2 \text{ (if } k>0)$$
$$> k+14$$
$$> k+8.$$

So $e^{k+1} > k+8$ $\boxtimes$

b) Base case $7^2 = 49$

$6(7)+2 = 44$

$49 > 44$ ✓

Suppose $k^2 > 6k+2$ for some $k \geq 7$.

Goal: $(k+1)^2 > 6(k+1)+2 = 6k+8$

Then $(k+1)^2 = k^2 + 2k+1 > (6k+2)+2k+1$
$$= (8k+1)+2 = (8k-5)+8$$
$$> 6k+8.$$

$8k-5 > 6k$ because $2k > 5$ because $k \geq 7$.

so $(k+1)^2 > 6k+8$ $\boxtimes$

③ Base case: $n=3$. In a triangle the angles of a triangle sum to 180 degrees.

$(n-2)(180) = (3-2)180 = 180$ so the formula agrees.

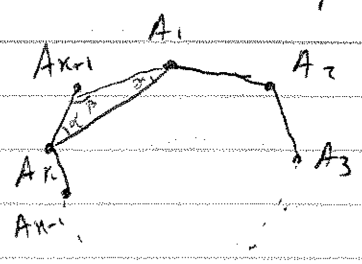
Induction Hypothesis: Suppose for a convex k-gon the sum of the angles is $(k-2)180$.

Goal: Prove that for a convex $(k+1)$-gon the sum of the angles is $(k-1)(180)$.

Proof of induction step:

Consider a convex $(k+1)$-gon. ($k+1$-sided polygon)

Suppose the vertices of the polygon are $A_1, A_2, \ldots, A_k, A_{k+1}$



Draw the line segment connecting A_1 to A_k

Then $A_1, A_2, \ldots, A_k$ are the vertices of a convex k-gon so the sum o its angles is $(k-2)180$ by the induction hypothesis.

The sum of the angles of the $(k+1)$-gon are $\alpha + \beta + \gamma$ + sum of angles of k-gon.

Since A_1, A_k, A_{k+1} is a triangle and the angles of a triangle sum to 180 degrees, $\alpha + \beta + \gamma = 180$.

Therefore the sum of the angles of the $(k+1)$-gon is
$(k-2)180 + 180 = 180((k-2)+1) = 180(k-1)$ $\boxtimes$

(4) a) For the sake of contradiction suppose $\sqrt{2}$ is rational.

b) For the sake of contradiction suppose $a > 1$ but $a^2 \leq \sqrt{a}$.

c) For the sake of contradiction suppose there exists a real number x such that $x^2 < 0$.

d) For the sake of contradiction suppose $4 | n$ and $4 | n+2$.

(5) Let's prove it by contradiction.
Suppose $x \in \mathbb{R}$ and $x^2 < 0$.
Let's do it in three cases.

Case 1: Suppose $x > 0$. Since $x > 0$ and $x > 0$
then $(x)(x) > (0)(0)$
so $x^2 > 0$. $\Rightarrow\Leftarrow$

Case 2: Suppose $x < 0$. Since $x < 0$ and $x < 0$
then $x(x) > 0(0)$
so $x^2 > 0$. $\Rightarrow\Leftarrow$

Case 3: Suppose $x = 0$, then $x^2 = 0$ so $x^2 \geq 0$. $\Rightarrow\Leftarrow$

In all cases $x^2 \geq 0$ so we reach a contradiction so $x^2 \geq 0$.

(6) a) Reflexive, Symmetric, Anti-symmetric, Transitive

b) Reflexive, Anti-symmetric, Transitive

c) Symmetric, Transitive

d) Antisymmetric, Transitive

(7) Proof:

- R is reflesive because a triangle T has the same angles as itself, so T is related to T.

- R is symmetric because if T R S (for triangles T and S) then the angles of T equal the angles of S. So the angles of S equal the angles of T. So S R T, so R is symmetric.

- Transitive: Suppose T, S, V are triangles such that

  T R S and S R V.

  Since T R S the angles of T equal the angles of S.
  Since S R V the angles of S equal the angles of V.
  Therefore the angles of T equal the angles of V.
  So T R V.
  So R is transitive.

(8) a) $1 R 1$ so $1 \in [1]$
$1 R 2$ so $2 \in [1]$

so $[1] = \{1, 2\}$.

b) $4 R 4$ so $[4] = \{4\}$.

c) The people with the same parents as me are my siblings and myself, so that's my equivalence class.

d) The numbers between 100 and 200 with the same ten-digits as 123 are

$\boxed{[123] = \{120, 121, 122, 123, 124, 125, 126, 127, 128, 129\}}$

~~[123] = {103, 113, 123, 133, 14~~ ~~129~~