Math 499 – D. Yuen

Homework: Chapter 9

Note that when you have an infinite product, it is possible for the infinite product to be zero even when each factor is not zero. The point of Exercises 1 and 2 below is to rigorously show that Euler's infinite product $\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right)$ only has zeros where Euler claims.

1. Note that when you have an infinite product, it is possible for the infinite product to be zero even when every factor is not zero. Prove that the following infinite product is zero even though each factor is not zero: $\prod_{k=1}^{\infty}\left(1-\frac{1}{k+1}\right)$. That is, prove $\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\frac{1}{k+1}\right)=0$. Hint: Write out $\prod_{k=1}^{n}\left(1-\frac{1}{k+1}\right)$ and see that it simplifies.

2. (a) Find positive constants A, B such that
   $-Ax \leq \ln(1-x) \leq -Bx$ for all $0 < x < \frac{1}{2}$.
   Hint: study the graph of $\ln(1-x)$, but use first and second derivatives to prove your claim.

   (b) Let $\{a_k\}$ be a sequence of positive numbers such that $\sum_{k=1}^{\infty} a_k$ converges.
   Note that this implies that the sequence $\{a_k\}$ converges to 0.
   Prove that $\lim_{n\to\infty}\prod_{k=1}^{n}(1-a_k)$ exists.
   Hint: Since the $\{a_k\}$ converge to 0, there exists a big enough N such that $a_k < \frac{1}{2}$ for all $k \geq N$. So to prove the above, we only need to prove that $\lim_{n\to\infty}\prod_{k=N}^{n}(1-a_k)$ exists. For convenience, call $b_n = \prod_{k=N}^{n}(1-a_k)$ for $n \geq N$. What can we say about b_n? Apply some real analysis theorem on sequences to say that b_n must converge (such as, "An increasing (or decreasing) bounded sequence must converge.").

   (c) Continuiung from (b), assume further that $a_k \neq 1$ for all k. Use part (a) to prove that $\lim_{n\to\infty}\prod_{k=1}^{n}(1-a_k)$ converges to a nonzero number.

   Hint: using the same N from (b), we only need to prove that $\lim_{n\to\infty}\prod_{k=N}^{n}(1-a_k)$ is nonzero. Investigate the sequence of $\ln b_n$ and use part (a).

   (d) What does this all say about Euler's infinite product? Does it converge? For what values of x is Euler's infinite product $\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right)$ zero?