Homework 2: Math 311: Intro to Real Analysis

Problem 1. (**Exercise 2.3.1 and 2.3.2**)

(a) Find a value of n so that any partial sum with at least n terms is within $1/100$ of the target value $\pi/4$ of the series $1 - 1/3 + 1/5 - 1/7 + \cdots$. Justify your answer.

(b) Find a value of n so that any partial sum with at least n terms is within $1/1000$ of the target value of the series expansion of $\arctan(1/2)$. Justify your answer.

Problem 2. (**Exercise 2.3.4 and 2.3.5**)

(a) Prove that

$$\frac{\pi}{8} = \frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \cdots .$$

Hint: Use that $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots$.

(b) Prove Machin's identity:

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$

Problem 3. (**Exercise 2.3.11**) It may appear that Newton's binomial series can only be used to find approximations to square roots of numbers between 0 and 2, but once you can do this, you can find a series for the square root of any positive number. If $x > 2$, then find an integer n so that $n^2 \le x < (n+1)^2$. It follows that $\sqrt{x} = n\sqrt{x/n^2}$ and $1 \le x/n^2 < 2$.

(a) Use this idea to find a series expansion for $\sqrt{13}$.

(b) Find a value of n so that any partial sum with at least n terms is within 0.001 of the target value $\sqrt{13}$. Justify your answer.

Problem 4. (**Exercise 2.4.1 and 2.4.2**)

(a) Give an example of a series that diverges to ∞ but whose partial sums do not form an increasing sequence.

(b) Give an example of a series that does not diverge to ∞ but whose partial sums are increasing.

Problem 5. (**Exercise 2.4.7**)

(a) Express the follows series in terms of π and $\ln(2)$:

$$1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots .$$

(b) Check your result by evaluating the first 1000 terms of this series.

Problem 6. (**Exercise 2.4.9**) We've proved that

$$\gamma = \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m^2} - \frac{1}{3}\sum_{m=1}^{\infty}\frac{1}{m^3} + \frac{1}{4}\sum_{m=1}^{\infty}\frac{1}{m^4} - \frac{1}{5}\sum_{m=1}^{\infty}\frac{1}{m^5} + \cdots . \tag{1}$$

The zeta (greek letter ζ) function is defined as

$$\zeta(k) = \sum_{m=1}^{\infty}\frac{1}{m^k}.$$

Euler famously proved that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and in general, that if k is even, then $\zeta(k)$ is a rational times π^k. There is no simple formula when k is odd, but these values of $\zeta(k)$ are also known to very high accuracy.

(a) What happens to $\zeta(k)$ as k increases?

(b) Assuming that we have arbitrarily good accuracy on the values of $\zeta(k)$, how many terms of the series (1) are needed to calculate γ to within 10^{-6}?

Problem 7. (**Exercise 2.4.10**) We know that the harmonic series does not converge. A result that is often seen as surprising is that if we eliminate those integers that contain the digit 9, the partial sums of the resulting series do stay bounded. Prove that the partial sums of the reciprocals of the integers that do not contain any 9's in their decimal representation are bounded.

Figure 1: Funny cartoon related to problem 7.

Problem 8. (**Exercise 2.4.16 and 2.4.17**)

(a) Use the fact that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \ln(n) - \gamma > -\frac{1}{2}\sum_{m=n}^{\infty}\frac{1}{m^2}$$

to find a lower bound for $1 + 1/2 + 1/3 + \cdots + 1/(n-1)$ in the form $\ln(n) + \gamma - R(n)$, where $R(n)$ is a **rational function** of n (a ratio of polynomials).

(b) Show that $\ln(n) + \gamma - R(n) > \ln(n-1) + \gamma$.

(c) Use (a) and (b) to find the precise smallest integer n such that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > 100.$$

Show the work that leads to your answer.