Catalan Lecture
Math 499: Senior Sem

August 30, 2017

Problem 1. Let n be a positive integer. Consider the square grid whose endpoints are $A(0,0)$, $(n,0)$, $B(n,n)$, $(0,n)$. Prove that the number of paths from $(1,0)$ to (n,n) that don't cross the diagonal AB (the path is allowed to touch the diagonal, but not cross it) is

$$\frac{1}{n+1}\binom{2n}{n}.$$

Also show the number of paths from $(1,0)$ to (n,n) that don't touch the diagonal AB is

$$\frac{1}{n}\binom{2n-2}{n-1}.$$

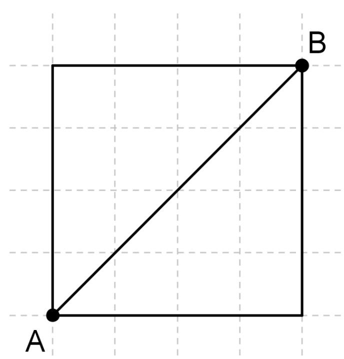

Figure 1: Catalan Diagram.

Solution 1. Let's consider the first part. Let C_n be the number of paths[1]. Fix $k \in \{0, 1, 2, \ldots, n-1\}$. We'll count the number of paths that satisfy that k is the largest number less than n where the path touches (k,k). The number of paths to reach (k,k) is C_k. The next step after (k,k) has to be to $(k+1,k)$. From there, the path to (n,n) cannot touch the diagonal again, therefore it must reach $(n,n-1)$. The paths from $E=(k+1,k)$ to $F=(n,n-1)$ don't cross the diagonal EF, because if they would, they would touch the AB line again. But the number of paths from E to F in such conditions resemble the original problem after translating everything by $(k+1,k)$ to get $E'=(0,0)$ and $F'=(n-k-1,n-k-1)$. Hence the number of paths from (k,k) to (n,n) is C_{n-k-1}. Note that the second part of the problem is the case $k=0$, so the answer to part 2 is C_{n-1}. If we prove $C_n = \frac{1}{n+1}\binom{2n}{n}$, then we prove both parts. Going back to our analysis: given k, the number of paths that last touch the diagonal at (k,k) is C_kC_{n-k-1}. After adding for all values of $k \in \{0,1,2,\ldots,n-1\}$, we get the following identity:

$$C_n = C_0C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \ldots + C_{n-1}C_0 = \sum_{k=0}^{n-1} C_kC_{n-k}. \qquad (1)$$

[1] We use this notation because these numbers are called Catalan numbers.

We know $C_0 = 1$. Then $C_1 = C_0C_0 = 1$, so $C_2 = C_0C_1 + C_0C_1 = 2$. Then $C_3 = C_0C_2 + C_1C_1 + C_2C_0 = 2 + 1 + 2 = 5$. Note that

$$C_0 = 1 = \frac{1}{1}\binom{0}{0}, \quad C_1 = 1 = \frac{1}{2}\binom{2}{1}, \quad C_2 = 2 = \frac{1}{3}\binom{4}{2}, \quad C_3 = 5 = \frac{1}{4}\binom{6}{3}.$$

We'll proceed with a technique known as "generating functions". Let $c(x)$ be the following polynomial:

$$c(x) = \sum_{n=0}^{\infty} C_n x^n. \tag{2}$$

Now, use that $C_0 = 1$ and (1), and plug into (2) to get

$$\begin{aligned}
c(x) &= 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} C_k C_{n-k-1} x^n \\
&= 1 + x\sum_{n=1}^{\infty}\sum_{k=0}^{n-1} C_k C_{n-k-1} x^{n-1} \\
&= 1 + x\sum_{n=0}^{\infty}\sum_{k=0}^{n} C_k C_{n-k} x^n.
\end{aligned}$$

The last equality is due to changing the variable n to $n-1$. Now change the order of the summands. To do this note that k ranges from 0 to ∞, but then $n \geq k$, so

$$\begin{aligned}
c(x) &= 1 + x\sum_{k=0}^{\infty}\sum_{n=k}^{\infty} C_k C_{n-k} x^n \\
&= 1 + x\sum_{k=0}^{\infty} C_k \sum_{n=k}^{\infty} C_{n-k} x^n \\
&= 1 + x\sum_{k=0}^{\infty} C_k x^k \sum_{n=k}^{\infty} C_{n-k} x^{n-k} \\
&= 1 + x\sum_{k=0}^{\infty} C_k x^k \sum_{m=0}^{\infty} C_m x^m \\
&= 1 + x\left(\sum_{k=0}^{\infty} C_k x^k\right)\left(\sum_{m=0}^{\infty} C_m x^m\right) \\
&= 1 + x(c(x))^2.
\end{aligned}$$

Using the quadratic formula, we get that

$$c(x) = \frac{1 \pm \sqrt{1-4x}}{2x}. \tag{3}$$

There are two candidates for $c(x)$ because of the $\pm$. Let's figure out which one is the correct one. Note that because the total number of paths from A to B is $\binom{2n}{n}$, then $C_n \leq \binom{2n}{n}$. But $\binom{2n}{n} \leq 4^n$ because the sum of the binomial coefficients in the $2n$-th row of Pascal's triangle is $2^{2n} = 4^n$. Using that $C_0 = 1$, and that $C_n \leq 4^n$, we get

$$c\left(\frac{1}{8}\right) = \sum_{n=0}^{\infty} C_n \left(\frac{1}{8}\right)^n \leq 1 + \sum_{n=1}^{\infty} 4^n \left(\frac{1}{8}\right)^n = 1 + 1 = 2.$$

But

$$\frac{1 + \sqrt{1 - \frac{4}{8}}}{2\left(\frac{1}{8}\right)} > 4 > 2 \geq c\left(\frac{1}{8}\right).$$

Therefore $c(x) = \frac{1-\sqrt{1-4x}}{2x}$.

Now, consider the following theorem:

Theorem 1 (Newton's Generalized Binomial Theorem)**.** Let x and y be real numbers with $|x| > |y|$. Let r be another real number. Then

$$(x+y)^r = x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \dots,$$

where the sum has infinitely many terms. Note that when r is a positive integer, the coefficient of $x^{r-k}y^k$ is 0 for all $k > r$, and it matches the well-known Binomial Theorem.

In particular, if $x = 1$ and $r = 1/2$, we have

$$\sqrt{1+y} = 1 + \frac{1}{2}y + \frac{1(-1)}{2^2 \cdot 2!}y^2 + \frac{1(-1)(-3)}{2^3 \cdot 3!}y^3 + \dots$$

The coefficient of y^n (for $n \geq 1$) looks like

$$\begin{aligned}
\frac{1(-1)(-3)\cdots(-(2n-3))}{2^n \cdot n!} &= (-1)^{n-1}\frac{(2n-3)(2n-5)\cdots(3)(1)}{2^n \cdot n!} \\
&= (-1)^{n-1}\frac{(2n-2)!}{2^n \cdot n! \cdot (2n-2)(2n-4)\cdots(4)(2)} \\
&= (-1)^{n-1}\frac{(2n-2)!}{2^n \cdot n! \cdot 2^{n-1}(n-1)!} \\
&= (-1)^{n-1}\left(\frac{2}{n \cdot 4^n}\right)\binom{2n-2}{n-1}.
\end{aligned}$$

Now, let $y = -4x$. Then we have

$$\begin{aligned}
\sqrt{1-4x} &= 1 + \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2}{n \cdot 4^n}\right)\binom{2n-2}{n-1}(-4x)^n \\
&= 1 - 2\sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^n.
\end{aligned}$$

Therefore

$$\begin{aligned}
c(x) &= \frac{1-\sqrt{1-4x}}{2x} \\
&= \frac{1-\left(1-2\sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^n\right)}{2x} \\
&= \sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^{n-1} \\
&= \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n.
\end{aligned}$$

By the definition of $c(x)$ we can now conclude that

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Alternative Solution : The following beautiful solution is due to the 19th century French mathematician, Désiré André [**?**].

Consider all paths from $(0,0)$ to (n,n) regardless of whether they satisfy the conditions. We'll call the paths that go over the diagonal "bad paths". We will count the number of bad paths in a clever way. The

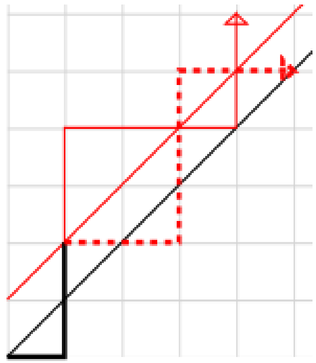

Figure 2: The invalid portion of the path is flipped. The original path is after $(k, k+1)$ is drawn in red dashes, while the new path is drawn in solid red. Note how the new path ends at $(n-1, n+1)$ instead of (n, n). This figure was created by Javalenok (Own work) [CC BY-SA 4.0 (`http://creativecommons.org/licenses/by-sa/4.0`)], via Wikimedia Commons.

idea is to consider the diagonal GH where $G = (0, 1)$, and $H = (n-1, n)$. If a path is bad, it must intersect this diagonal. Consider the first intersection in the path. Then reflect versus AB everything beyond this intersection as in Figure 2.

In a path from A to B there must be n horizontal steps and n vertical steps. Let $(k, k+1)$ be the first time you intersect GH (in a bad path). Then you have taken k horizontal steps and $k+1$ vertical steps. So the rest of the path would have $n-k$ horizontal steps and $n-k-1$ vertical steps. The action of reflection changes the horizontal steps to vertical and viceversa. Therefore, after $(k+1, k)$, the new path has $n-k$ vertical steps and $n-k-1$ horizontal steps. Overall you take $k + (n-k-1) = n-1$ horizontal steps and $k+1+(n-k) = n+1$ vertical steps. Therefore you end up in $(n-1, n+1)$. So every "bad path" can be associated to a path from $(0, 0)$ to $(n-1, n+1)$. It goes both ways, since the steps are reversible. Indeed, consider a path from $(0, 0)$ to $(n-1, n+1)$. Let $(k, k+1)$ be the smallest k through which the path goes (it must intersect GH, so such a k exists). Then flip everything past this point. Now you have a "bad path" from $(0, 0)$ to (n, n). Therefore, the number of bad paths, is the number of paths from $(0, 0)$ to $(n-1, n+1)$. The number of such paths is

$$\binom{(n-1)+(n+1)}{n-1} = \binom{2n}{n-1}.$$

Since the number of total paths is $\binom{2n}{n}$, then

$$\begin{aligned} C_n = \binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} \\ &= \frac{(2n)!}{n!n!}\left(1 - \frac{n}{n+1}\right) = \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$