Math 499 – D. Yuen

Homework: Chapter 1

1. Assume that the following can readily be done in construction by compass and straightedge.
   - Construct a perpendicular to a given line through a given point (whether or not the point is on the line).
   - Construct a line parallel to a given line through a given point not on the given line.
   - Transfer a length. That is, given points A, B, C and a line ℓ through C, we can construct a point D on ℓ such that $AB = CD$. (This is done by opening up the compass to radius AB and drawing a circle centered at C.)
   - Bisect a line segment.

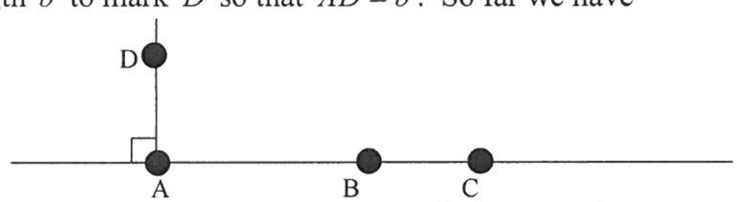
   In answering the questions below, give step by step instructions. You may use the above known constructions.

   Suppose we are given a line segment that is declared to have length one. Suppose we are given two line segments of lengths a and b.

   (a) Show how you would construct a line segment of length $a+b$.

   (b) Assuming $a < b$ for this part, show how you would construct a line segment of length $b-a$.

   (c) Assume $a > 1$ for this part. Construct the following figure. Draw any line and transfer the lengths a and 1 to this line so that we have points A, B, C on this line with $AB = 1$, $AC = a$ with B between A and C. Draw a perpendicular line at A, and transfer the length b to mark D so that $AD = b$. So far we have

   Now construct a line through C that is parallel to $\overleftrightarrow{BD}$. This new line intersects $\overleftrightarrow{AD}$ at say E. Prove that we constructed $AE = ab$. Hint: use similar triangles.

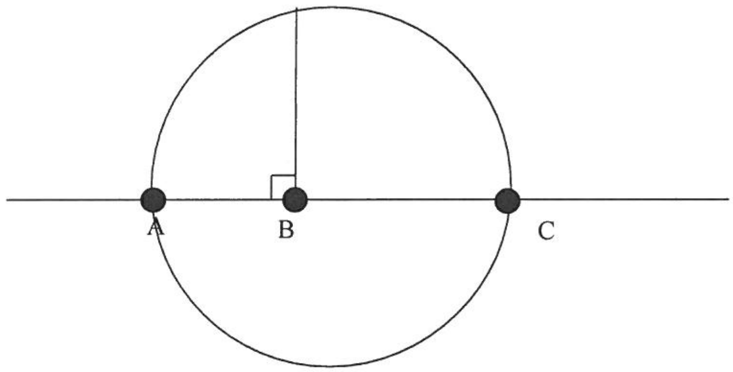
   (d) Assume $a < 1$ for this part. Show how to construct a line segment of length ab. Hint: modify (c).

   (e) Assume $a < b$ for this part. Show how to construct a line segment of length b/a. Hint: Do a variation on the construction in (c).

   (f) Assume $a > b$ for this part. Show how to construct a line segment of length b/a.

[continued on back]

(g) Perform the following construction. Draw any line and transfer the lengths a and 1 to this line so that we have points A, B, C on this line with $AB = 1$, $BC = a$ with B between A and C. Bisect the segment $\overline{AC}$ at D and draw a circle with D as center going through A and C. So $\overline{AC}$ is a diameter of the circle. Draw a perpendicular line at B and it intersects the circle at say E. So far we have



Prove that we have constructed $BE = \sqrt{a}$. Hint: use theorem that an angle inscribed in a semicircle is a right angle and then somehow use similar triangles.

2. The above problem proves that starting with a given unit length segment, we can construct any length that involves natural numbers and the operations $+$, $-$, $\times$, $\div$, and $\sqrt{\ }$. Given a line segment of length one, perform the following.
   (a) Construct a line segment of length $2/3$.
   (b) Construct a line segment of length $\sqrt{1+\sqrt{2}}$.

3. In fact, it can be proven that any length that can be constructed must come from the operations $+$, $-$, $\times$, $\div$, and $\sqrt{\ }$. We define a number to be <u>constructible</u> if it <u>can</u> be expressed as integers and operations $+$, $-$, $\times$, $\div$, and $\sqrt{\ }$. We must be cautious in that sometimes just because a number is not yet written in that form, it does not mean it is not constructible, because it might have another equivalent expression. For example, $\sqrt[3]{7+5\sqrt{2}}$ is not written in that form, but it turns out it is equal to $1+\sqrt{2}$, and so $\sqrt[3]{7+5\sqrt{2}}$ is constructible. For this question, assume you know that $\sqrt[3]{2}$ is not constructible (a fact proven in Math 331). Using this fact, prove that $\sqrt[3]{4}$ is not constructible. Hint: do a proof by contradiction.