Homework: Chapter 11

The goal of this homework is to prove that the set of algebraic numbers is denumerable. Recall that the definition of an algebraic number is:

Definition. We say $\alpha \in \mathbf{C}$ (complex number) is algebraic if there exists a polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ with $n \geq 1$ and **integers** $a_0, a_1, \ldots, a_n \in \mathbf{Z}$ with $a_n \neq 0$ such that $p(\alpha) = 0$. That is, if α is a root of some nonconstant integer coefficient polynomial. Let $\mathbf{A}$ denote the set of algebraic numbers. (It is actually a field.)

Definition. For any polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$,define its Garfield-norm to be $G(p(x)) = n + |a_0| + |a_1| + \cdots + |a_n|$. For example, the Garfield-norm of $3x\text{^}2 - 4x + 2$ is $G(3x\text{^}2 - 4x + 2) = 2 + 3 + 4 + 2 = 11$.

1. Prove that every nonconstant integer polynomial has a natural number as its Garfield-norm.

2. Let T be the set of all nonconstant integer coefficient polynomials. For each $k \in \mathbf{N}$, define
   $S_k = \{p(x) \in T : G(p(x)) = k\}$
   List the sets S_1, S_2, S_3.

3. Prove that each S_k is a finite set.

4. For each algebraic number $\alpha \in \mathbf{A}$, define its Odie-norm to be $Odie(\alpha) = \min\{G(p(x)) : p(x) \in T, p(\alpha) = 0\}$. It is the minimum Garfield-norm over all integer-coefficient polynomials that have α as a root. For example,

   $Odie(-5) = 7$ because $p(x) = x + 5$ is the smallest polynomial (in the sense of Garfield norms) such that $p(-5) = 0$.
   $Odie(\sqrt[3]{2}) = 6$ because $p(x) = x^3 - 2$ is the smallest polynomial such that $p(\sqrt[3]{2}) = 0$.

   Find the Odie-norms of the following algebraic numbers: (a) 17 (b) $3/17$ (c) $\sqrt{3}$.

5. For each $k \in \mathbf{N}$, define
   $W_k = \{\alpha \in \mathbf{A} : Odie(\alpha) = k\}$
   List the sets W_1, W_2, W_3.

6. Prove that each W_k is a finite set.

7. Prove that $\mathbf{A} = \bigcup_{k=1}^{\infty} W_k$.

8. Prove that $\mathbf{A}$ is denumerable.