Induction Proof Practice

1. Prove that for any positive integer \( n \),

\[
1 + 3 + 6 + \cdots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}.
\]

2. Prove that for any positive integer \( n \),

\[
2^n > n.
\]

3. Prove by induction that the number of subsets of a set with \( n \) elements is \( 2^n \).

4. Prove that every positive integer \( n > 1 \), has a prime divisor.

5. Evaluate the sum

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{999 \cdot 1000}.
\]

Solutions

1. \textit{Proof.} For \( n = 1 \), the left side is 1 and the right side is \( \frac{1 \cdot 2}{6} = 1 \).

Suppose the statement is true for \( n = k \), namely, suppose that for some \( k \geq 1 \), we have

\[
1 + 3 + \cdots + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}.
\]

Now, consider the case \( n = k + 1 \). We have

\[
1 + 3 + \cdots + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} = \left(1 + 3 + \cdots + \frac{k(k+1)}{2}\right) + \frac{(k+1)(k+2)}{2} = \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{6} + \frac{2}{6(k+3)} = \frac{(k+1)(k+2)}{6} \cdot \frac{3}{(k+3)} = \frac{(k+1)(k+2)(k+3)}{6}.
\]

Therefore, we’ve shown that when it’s true for \( k \), it implies it’s true for \( k + 1 \). We’ve finished the proof by induction.

\( \square \)

2. \textit{Proof.} The base case is \( n = 1 \) and we can see that \( 2^1 = 2 > 1 \). Therefore it’s true for \( n = 1 \).

Let’s assume that it’s true for \( n = k \), namely, suppose \( 2^k > k \). We have \( 2^{k+1} = 2 \cdot 2^k > 2 \cdot k \geq k + 1 \) whenever \( 2k \geq k + 1 \), which is true for \( k \geq 1 \). Therefore, \( 2^{k+1} > k + 1 \) and hence we’ve proved the general statement by induction.

\( \square \)
3. **Proof.** For \( n = 0 \), we have that the only subset of a set with zero elements is the empty set. Therefore, it has one subset. But \( 2^0 = 1 \), so the statement is true for \( n = 0 \). For \( n = 1 \), let \( A = \{a\} \) be our set with one element. Then the only subsets are \( \emptyset \) and \( \{a\} \). Therefore, it has two subsets. Since \( 2^1 = 2 \), we have that the statement to be proved is true for \( n = 1 \). We have our base case.

Now, for an integer \( k \geq 1 \), suppose that the number of subsets of a set with \( k \) elements is \( 2^k \). This will be the induction hypothesis.

Suppose \( A = \{a_1, a_2, \ldots, a_k, a_{k+1}\} \) is a subset with \( k+1 \) elements. Let’s consider all the subsets. Let \( T \) be the set of subsets of \( A \) that contain \( a_{k+1} \) and \( U \) be the set of subsets that don’t contain \( a_{k+1} \). Note that the set of subsets of \( A \) is the disjoint union of \( T \) and \( U \). We’re going to show that \( |T| = |U| = 2^k \). First, let’s consider \( U \). The subsets of \( A \) that don’t contain \( a_{k+1} \) are the subsets of \( \{a_1, a_2, \ldots, a_k\} \). By the induction hypothesis, there are \( 2^k \) of these. Now consider the subsets of \( A \) that contain \( a_{k+1} \). Once you ignore that term, the rest of the subset must be a subset of \( \{a_1, a_2, \ldots, a_k\} \), so by the induction hypothesis there are \( 2^k \) of these. Therefore, the number of subsets of \( A \) is \( |T| + |U| = 2^k + 2^k = 2^{k+1} \), which is what we wanted to prove.

\[ \square \]

4. **Proof.** For \( n = 2 \), the prime divisor is 2. Suppose that all numbers \( 1 < i \leq k \) have a prime factor. We want to show that \( k + 1 \) also has a prime factor. If \( k + 1 \) is prime, then it has a prime factor (itself). If \( k + 1 \) is not prime, then there exist \( a, b \) satisfying \( 1 < a \leq b < k + 1 \) such that \( k + 1 = ab \). But then \( 1 < a \leq k \). By the strong induction hypothesis, \( a \) has a prime factor \( p \). But then \( p|a \) and \( a|k + 1 \), so \( p|k + 1 \). Therefore \( k + 1 \) has a prime factor. Therefore, by strong induction, all integers greater than 1 have a prime factor.

\[ \square \]

5. Let’s find a pattern:

\[
\begin{align*}
\frac{1}{1 \cdot 2} &= \frac{1}{2} \\
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} &= \frac{2}{3} \\
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} &= \frac{3}{4} \\
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} &= \frac{4}{5}.
\end{align*}
\]

It seems the pattern is that the sum up to \( \frac{1}{(k-1)k} \) is \( \frac{k-1}{k} = 1 - \frac{1}{k} \). This suggests the answer to the question is \( \frac{999}{1000} \). Let’s prove that the pattern persists by using induction:

**Proof.** The base case are the examples listed above. As our induction hypothesis suppose

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k-1)k} = \frac{k-1}{k}.
\]
Now, consider the next term:

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k-1)k} + \frac{1}{k(k+1)} = \frac{k-1}{k} + \frac{1}{k(k+1)}
\]

\[
= \frac{1}{k(k+1)}((k-1)(k+1) + 1)
\]

\[
= \frac{1}{k(k+1)}(k^2 - 1 + 1)
\]

\[
= \frac{k}{k(k+1)}
\]

\[
= \frac{k}{k+1}.
\]

This completes the proof by induction. \(\square\)