Math 230 Midterm #1
September 30, 2013

Instructions: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

• You may NOT use a calculator.

• Show all of your work.

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Official Cheat Sheet

1. Let $A$ be a set. Then $2^A$ is the set of all subsets of $A$. For example, if $A = \{1, 2\}$, then $2^A = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

2. $|A|$ is the number of elements of $A$. A useful formula is: $|A \cup B| = |A| + |B| - |A \cap B|$ if $A$ and $B$ are finite sets. Another useful formula is $|2^A| = 2^{|A|}$ when $A$ is finite.

3. Here are some Boolean algebra properties (which can be translated easily to set properties by equating $\lor$ with $\cup$ and $\land$ with $\cap$):
   - $x \land y = y \land x$ and $x \lor y = y \lor x$.
   - $(x \land y) \land z = x \land (y \land z)$ and $(x \lor y) \lor z = x \lor (y \lor z)$.
   - $x \land (y \lor z) = (x \land y) \lor (x \land z)$ and $x \lor (y \land z) = (x \lor y) \land (x \lor z)$.

4. $\mathbb{Z}$ is the set of integers. $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of positive integers.

5. Let $A$ and $B$ be sets. Then
   - $A \cup B = \{x | x \in A \text{ or } x \in B\}$,
   - $A \cap B = \{x | x \in A \text{ and } x \in B\}$,
   - $A - B = \{x | x \in A \text{ and } x \notin B\}$,
   - $A \Delta B = (A - B) \cup (B - A)$,
   - $A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$.

6. Let $a$ and $b$ be integers.
   - $a$ is even if there exists an integer $c$ such that $a = 2c$.
   - $a$ is odd if there exists an integer $c$ such that $a = 2c + 1$.
   - We say $a | b$ ($a$ divides $b$) if there exists an integer $c$ such that $b = ac$.
   - $a$ is composite if $|a| > 1$ and there exists $c$ such that $1 < c < |a|$ and $c | a$.
   - $a$ is prime if $a > 1$ and $a$ is not composite.
   - $a$ is perfect if $a$ equals the sum of its positive divisors less than $a$. 
1. True or False (Just answer true or false, you don’t need to explain your answer):

(a) [2 points] -23 is prime.

(b) [2 points] 7|1001.

(c) [2 points] The sum of two odd numbers is odd.

(d) [2 points] T ⊆ A if and only if T ∈ 2^A.

(e) [2 points] ∅ ⊆ {∅}.

(f) [2 points] Let \( n = 2^{p-1}(2^p - 1) \) where \( 2^p - 1 \) is prime. \( n \) is a perfect number.

(g) [2 points] 2 ∈ \{1, 2, {1, 2}\}.

(h) [2 points] If \( x^2 < 0 \), then \( x \) is a perfect number.

(i) [2 points] Two right triangles that have hypotenuses of the same length have the same area.

(j) [2 points] \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = 1 \).
2. For the following pairs of statements \( A, B \), write \( a \) if the statement “If \( A \), then \( B \)” is true, write \( b \) if the statement “If \( B \), then \( A \)” is true and write \( c \) if the statement ”\( A \) if and only if \( B \)” is true. You should write all that apply.

(a) [5 points] \( A: x > 0. \) \( B: x^2 > 0. \)

(b) [5 points] \( A: \) Ellen is a grandmother. \( B: \) Ellen is female.

(c) [5 points] \( A: x \) is odd. \( B: x + 1 \) is even.

(d) [5 points] \( A: \) Polygon \( PQRS \) is a rectangle. \( B: \) Polygon \( PQRS \) is a square.
3. Proofs:

(a) [5 points] Using the definition of odd integer provided in the “cheat sheet”, prove that if \( n \) is an odd integer, then \(-n\) is also an odd integer.

(b) [5 points] Let \( a, b \) and \( d \) be integers. Suppose \( b = aq + r \) where \( q \) and \( r \) are integers. Prove that if \( d|a \) and \( d|b \), then \( d|r \).
4. Find counterexamples to disprove the following statements:
   
   (a) [5 points] An integer $x$ is positive if and only if $x + 1$ is positive.

   (b) [5 points] An integer is a *palindrome* if it reads the same forwards and backwards when expressed in base 10. For example, 1331 is a *palindrome*. All *palindromes* are divisible by 11.

   (c) [5 points] If $a$, $b$ and $c$ are positive integers then $a^{(b^c)} = (a^b)^c$.

   (d) [5 points] Let $A$, $B$ and $C$ be sets. Then $A - (B - C) = (A - B) - C$. 
5. Boolean Algebra

(a) [5 points] Prove or disprove that the Boolean expressions \( x \to \neg y \) and \( \neg(x \to y) \) are logically equivalent.

(b) [5 points] The expression \( x \to y \) can be rewritten in terms of only the basic operations \( \land, \lor \) and \( \neg \); that is, \( x \to y = (\neg x) \lor y \). Find an expression that is logically equivalent to \( x \leftrightarrow y \) that uses only the operations \( \land, \lor, \neg \) and prove that your expression is correct.
6. Consider the following proposition. Let $N$ be a two-digit number and let $M$ be the number formed from $N$ by reversing the digits of $N$. Now compare $N^2$ and $M^2$. The digits of $M^2$ are precisely those of $N^2$, but reversed. For example:

$$
10^2 = 100 \quad \quad 01^2 = 001 \\
11^2 = 121 \quad \quad 11^2 = 121 \\
12^2 = 144 \quad \quad 21^2 = 441 \\
13^2 = 169 \quad \quad 31^2 = 961
$$

and so on. Here is a proof of the proposition:

**Proof.** Since $N$ is a two-digit number, we can write $N = 10a + b$ where $a$ and $b$ are the digits of $N$. Since $M$ is formed from $N$ by reversing digits, $M = 10b + a$.

Note that $N^2 = (10a + b)^2 = 100a^2 + 20ab + b^2 = (a^2) \times 100 + (2ab) \times 10 + (b^2) \times 1$, so the digits of $N^2$ are, in order, $a^2, 2ab, b^2$.

Likewise, $M^2 = (10b + a)^2 = (b^2) \times 100 + (2ab) \times 10 + (a^2) \times 1$, so the digits of $M^2$ are, in order, $b^2, 2ab, a^2$, exactly the reverse of $N^2$, which completes the proof.

(a) [5 points] Prove that the proposition is false.

(b) [5 points] Explain why the proof is invalid.
7. Counting

(a) [5 points] In how many ways can we make a list of three integers \((a, b, c)\) where \(0 \leq a, b, c \leq 9\) such that \(a + b + c\) is even?

(b) [5 points] Evaluate \(\prod_{k=0}^{100} \frac{k^2}{k + 1}\).
8. Let $A \times B = \{(1, 2), (1, 3), (1, 7), (2, 2), (2, 3), (2, 7)\}$.

(a) [5 points] What is $A \cup B$?

(b) [5 points] What is $A \cap B$?

(c) [5 points] What is $A - B$?

(d) [5 points] What is $A \Delta B$?
9. [10 points] Let $A, B$ and $C$ be sets. Prove that

$$(A \cup B) - C = (A - C) \cup (B - C).$$