1) a) T  
   \[ 10 \mid 3 - 23 = -20 \]

   b) F  
   \[ 7 + 95 - 111 = -16 \]

   c) T  
   \[ a < b \land b < c \implies a < c \]

   d) F  
   \[ R = \{ (1,1), (2,2), (3,3) \} \text{ on } \{1,2,3\} \text{ is both} \]

   e) T  
   \[ \text{Only } R \neq 0 \text{ then you could be both} \]

   f) F  
   \[ \text{Problem 5a is an example of a reflexive and symmetric relation that is not transitive.} \]

   g) T  
   \[ (\text{Proved in class}) \]

   h) T  
   \[ (\text{It's the base case}) \]

   i) T  
   \[ (\text{The difference is in the inductive hypothesis}) \]

   j) F  

2) a) Base case: \( n = 1 \)
   \[ 1 + 2 = 3 \]
   \[ 2^n - 1 = 3 \]

   So it's true.

   Assume \[ 1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1 \]

   Base: \[ 1 + 2 + 2^2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1 \]

   \[ 1 + 2 + 2^2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} \]

   \[ = 2 \cdot 2^{k+1} - 1 \]

   \[ = 2^{k+2} - 1, \]

   Hence it's true by induction.
b) For \( n = 1 \) we have \( 3 = 2(1)^2 + 1 = 3 \) \( \checkmark \)

Suppose \( 3 + 7 + 11 + \ldots + (4k-1) = 2k^2 + k \)

then \( 3 + 7 + 11 + \ldots + (4k-1) + (4k+3) = 2k^2 + k + 4k + 3 \)
\[ = 2k^2 + 4k + 2 + (k+1) = 2(k+1)^2 + (k+1) = 2(k+1)^2 + (k+1) \]
\[ = 2k^2 + 5k + 3 \]

(c) Base case: \( n = 3 \). \( 9 = 3^2 > 2(3) + 1 = 7 \) \( \checkmark \)

Suppose \( k^2 > 2k+1 \)

then \( (k+1)^2 = k^2 + 2k + 1 > (2k+1) + 2k + 1 = 4k + 2 \)

Since \( k \geq 3 \), \( 2k \geq 6 \)

So \( 4k + 2 > 2(k+1) + 1 \) so the statement is true for all \( n \geq 3 \) \( \checkmark \)

d) \( 2^5 = 32 > 25 = 5^2 \) so it's true for \( n = 5 \).

Suppose \( 2^k > k^2 \).

Then \( 2^{k+1} = 2 \cdot 2^k > 2k^2 = k^2 + k^2 \)

We want to show \( 2k^2 > (k+1)^2 = k^2 + 2k + 1 \)

so we want to show \( k^2 > 2k + 1 \),

but from (c) we know \( k^2 > 2k + 1 \) for \( k \geq 3 \)

Since \( k \geq 3 \), \( k^2 > 2k + 1 \) so \( 2k^2 > (k+1)^2 \)

So \( 2^{k+1} > (k+1)^2 \) \( \checkmark \)
a) Suppose there exist 4 consecutive integers whose sum is divisible by 4.

b) Suppose there is a point in P that is not in a common line with the rest.

c) For the sake of contradiction, suppose there exist 4 consecutive integers whose sum is not divisible by 4.

Suppose the 4 consecutive integers are \( n, n+1, n+2, n+3 \).

So \( 4 \mid n + (n+1) + (n+2) + (n+3) = 4n + 6 \)

\[ 4n + 6 = 4(n+1) + 2 \]

Since \( 4 \mid 4n + 6 \), there exists \( k \) s.t. \( 4n + 6 = 4k \).

So \( 6 = 4(k - n) \)

So \( 3 = k - n \in \mathbb{Z} \)

But \( k - n \notin \mathbb{Z} \). Therefore our assumption must be false, hence there do not exist 4 consecutive integers whose sum is divisible by 4.

b) Suppose not all points are lying in the same line.

Take 2 points. If all points lied on the same line, the line would include those 2 points. Take a point not on that line. Since not all points lie in the same line there is a point not on the same line as these 2 points.
But then those 2 points together with the point not on the line between those two points are 3 points that are not collinear. This contradicts our assumption that any 3 points are collinear.

5) a) \[ \begin{array}{c}
\text{Reflexive} \\
\text{Symmetric}
\end{array} \]

\[(2, 1) \nsim (1, 4) \Rightarrow (2, 4) \not\in R \]

Not transitive

Not irreflexive since \((1, 1) \in R\)

Not anti-symmetric since \((1, 2) \in R, (2, 1) \not\in R\)

b) \[ \begin{array}{c}
\text{Reflexive} \\
\text{Transitive} \\
\text{Antisymmetric}
\end{array} \]

Not irreflexive

Not symmetric

6) a) \([4] = \{1, 2, 3\}\]

\([1] = \{1, 2, 4\}\]

b) \([1, 2, 3] = \{1, 2, 3, 4, 5\} \cup \{1, 2, 5\} \cup \{2, 3, 5\} \cup \{2, 4, 5\}
\cup \{3, 4, 5\} \cup \{1, 3, 5\} \cup \{1, 4, 5\} \cup \{3, 4, 5\}\)
c) \[ \{ -3 \} = \{ -3, 3 \} \]
\[ \{ 0 \} = \{ 0 \} \]

d) \[ \{ (0, 1) \} = \{ (a, b) : a, b \in \mathbb{R}, a^2 + b^2 = 0^2 + 1^2 \} = \{ (x, y) : x, y \in \mathbb{R}, x^2 + y^2 = 1 \} \]

It's the unit circle (Circle of radius 1 centered at the origin).

7. Let's start with reflexive.

\((a, b) \mathcal{R} (a, b) \) because \( a^2 + b^2 = a^2 + b^2 \)

Hence \( \mathcal{R} \) is reflexive.

**Symmetric:**

Suppose \((a, b) \mathcal{R} (c, d)\) then \( a^2 + b^2 = c^2 + d^2 \)

so \( c^2 + d^2 = a^2 + b^2 \)

so \((c, d) \mathcal{R} (a, b)\), so \( \mathcal{R} \) is symmetric

**Transitive**

Suppose \((a, b) \mathcal{R} (c, d)\) and \((c, d) \mathcal{R} (e, f)\)

Goal: Prove \((a, b) \mathcal{R} (e, f)\).

\((a, b) \mathcal{R} (c, d) \Rightarrow a^2 + b^2 = c^2 + d^2 \)

\((c, d) \mathcal{R} (e, f) \Rightarrow c^2 + d^2 = e^2 + f^2 \)

\( \Rightarrow a^2 + b^2 = e^2 + f^2 \)

\( \Rightarrow (a, b) \mathcal{R} (c, d) \Rightarrow \mathcal{R} \) is transitive

\( \Rightarrow \mathcal{R} \) is an equivalence relation.