MATH 230 MIDTERM #2
March 3, 2014

INSTRUCTIONS: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- You may NOT use a calculator.
- Show all of your work.

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
<th>Score</th>
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<tr>
<td>1</td>
<td>20</td>
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<td><strong>Total:</strong></td>
<td><strong>110</strong></td>
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1. True or False (Just answer true or false, you don’t need to explain your answer):

(a) [2 points] Let \( R \) be an equivalence relation on the set \( A \). Let \( a \in A \), then \( a \in [a] \).
\[ \text{TRUE} \]

(b) [2 points] Let \( R \) be an equivalence relation on \( A \). If \( a \in A \) and \( b \in A \), then the intersection of \([a]\) and \([b]\) is empty.
\[ \text{FALSE} \text{ , } \forall a \neq b \text{ then } [a] \cap [b] = \emptyset. \]

(c) [2 points] Let \( R \) be a symmetric relation. Then \( R = R^{-1} \).
\[ \text{TRUE} \]

(d) [2 points] If a relation is symmetric then it is not antisymmetric.
\[ \text{FALSE} \text{ } \left( \text{"is-equal-to" relation is both} \right) \]

(e) [2 points] There is a relation that is reflexive and irreflexive.
\[ \text{TRUE} \text{ } \left( \text{the empty relation on the empty set} \right) \]

(f) [2 points] If a relation is reflexive and symmetric it must also be transitive.
\[ \text{FALSE} \]

(g) [2 points] A relation \( R \) on a set \( A \) is antisymmetric if and only if \( R \cap R^{-1} \subseteq \{(a, a) : a \in A\} \)
\[ \text{TRUE} \]

(h) [2 points] Let \( R \) be an equivalence relation on a set \( A \). The equivalence classes of \( R \) form a partition of the set \( A \).
\[ \text{TRUE} \text{ } \left( \text{by definition} \right) \]

(i) [2 points] The difference between a proof by strong induction and a proof by induction is that the base cases are dealt with differently.
\[ \text{FALSE} \text{ } \left( \text{the difference is in the hypothesis} \right) \]

(j) [2 points] The is-less-than-or-equal-to relation is an equivalence relation.
\[ \text{FALSE} \text{ } \left( \text{not symmetric} \right) \]
2. Prove that the following are true:

(a) [5 points] \(1 + 3 + 3^2 + \ldots + 3^n = \frac{3^{n+1} - 1}{2}\), for all positive integers \(n\).

\[
\text{For } n = 1: \quad 1 + 3 = 4 \quad \text{and} \quad \frac{3^2 - 1}{2} = 4 \quad \text{so it's true for } n = 1.
\]

\[
\text{Suppose } 1 + 3 + \ldots + 3^k = \frac{3^{k+1} - 1}{2}, \quad k
\]

\[
\text{Now } 1 + 3 + \ldots + 3^k + 3^{k+1} = \frac{3^{k+1} - 1}{2} + 3^{k+1} = \frac{3^{k+1} + 2 \cdot 3^{k+1}}{2} = \frac{3^{k+2} - 1}{2}.
\]

\[
\text{Since } 1 + 3 + \ldots + 3^k = \frac{3^{k+1} - 1}{2}, \quad \text{then the statement is true by induction.}
\]

(b) [5 points] \(1 + 5 + 9 + \ldots + (4n - 3) = 2n^2 - n\), for all positive integers \(n\).

\[
\text{For } n = 1: \quad 1 = 2(1^2) - 1 \equiv 1 \quad \checkmark
\]

\[
\text{Suppose } 1 + 5 + 9 + \ldots + (4k - 3) = 2k^2 - k.
\]

\[
\text{Prove } 1 + 5 + 9 + \ldots + (4k - 3) + (4k + 1) = 2(k+1)^2 - (k + 1)
\]

\[
= 2(k^2 + 2k + 1) - k - 1
\]

\[
= 2k^2 + 3k + 1.
\]

\[
(1 + 5 + 9 + \ldots + (4k - 3)) + (4k + 1) = 2k^2 - k + (4k + 1)
\]

\[
= 2k^2 + 3k + 1
\]

So the LHS equals the RHS. The proof is complete.
(c) [5 points] \( n^2 > 2n + 1 \), for all integers \( n \geq 3 \).

For \( n = 3 \), the LHS is 9 and the RHS is 7. \( 9 > 7 \) \( \checkmark \)

Suppose for some \( k \geq 3 \) \( k^2 > 2k + 1 \).

Let's prove \( (k+1)^2 > 2(k+1)+1 = 2k+3 \)

\((k+1)^2 = k^2 + 2k + 1 > (2k) + 2k + 1 = 4k + 2 \)

We want \( 4k + 2 > 2k + 3 \) so we need \( 2k > 1 \)

but \( 2k \geq 1 \) since \( k \geq 3 \).

Therefore \( (k+1)^2 > 2k + 3 \) and the proof is complete.

(d) [5 points] \( 2^n > n^2 \), for all integers \( n \geq 5 \).

Base Case: \( 2^5 = 32 > 25 = 5^2 \).

Suppose \( 2^k > k^2 \) for some \( k \geq 5 \).

\( 2^{k+1} = 2 \cdot 2^k > 2k^2 = k^2 + k^2 \).

We want to prove \( 2^{k+1} > (k+1)^2 = k^2 + 2k + 1 \).

So it comes down to showing that \( k^2 + k^2 > k^2 + 2k + 1 \) so to showing that \( k^2 > 2k + 1 \) for \( k \geq 5 \).

From (c) we know \( k^2 > 2k + 1 \) for \( k \geq 3 \),

therefore \( k^2 > 2k + 1 \) for \( k \geq 5 \) so

\( 2^{k+1} > (k+1)^2 \) \( \checkmark \)
3. For each of the following relations defined on the set \{1, 2, 3, 4\} determine whether they are reflexive, irreflexive, symmetric, antisymmetric and/or transitive.

(a) [5 points]

\[ R = \{(1, 1), (2, 2), (3, 3), (1, 4), (4, 4), (1, 3), (4, 3)\}. \]

Reflexive, Antisymmetric, Transitive

(b) [5 points]

\[ R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (4, 4), (1, 4), (4, 1)\}. \]

Reflexive, Symmetric

Not transitive because \((4, 1) \in R \land (1, 2) \in R\) but \((4, 2) \notin R\).
4. For each equivalence relation below, find the requested equivalence classes.

(a) [5 points] $R = \{(x, y) : x, y \in \mathbb{Z} \text{ and } |x| = |y|\}$. Find $[-3]$ and $[0]$.

$$[-3] = \{-3, 3\}$$

$$[0] = \{0\}$$

(b) [5 points] $R$ is has-the-same-size-as relation on $2\{1, 2, 3, 4, 5\}$. Find $\{1, 2, 3\}$.

$$\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\},\}$$
(c) [5 points] \( R \) is the relation
\[
R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (4, 4), (1, 4), (4, 1), (2, 4), (4, 2)\}
\]
on the set \( \{1, 2, 3, 4\} \). Find [1] and [2].

\[
[1] = \{1, 2, 4\}
\]
\[
[2] = \{1, 2, 4\}
\]

(d) [5 points] \( R \) is the relation on the set of points on the plane where the point \((a, b)\) is related to the point \((c, d)\) if and only if \(a^2 + b^2 = c^2 + d^2\) (for all \(a, b, c, d\) real numbers). What figure in the plane does \([0, 1]\) represent?

The unit circle.
5. [15 points] Let $R$ be the relation on the set of points on the plane where the point $(a, b)$ is related to the point $(c, d)$ if and only if $a^2 + b^2 = c^2 + d^2$ (for all $a, b, c, d$ real numbers). Prove that $R$ is an equivalence relation on the set of points on the plane.

To prove it we must prove $R$ is reflexive, symmetric, and transitive.

**Reflexive:**

$(a, b) R (a, b)$ because $a^2 + b^2 = a^2 + b^2$

so $R$ is reflexive.

**Symmetric:** Suppose $(a, b) R (c, d)$.

Then $a^2 + b^2 = c^2 + d^2$

so $c^2 + d^2 = a^2 + b^2$

so $(c, d) R (a, b)$

so $R$ is symmetric.

**Transitive:** Suppose $(a, b) R (c, d)$ and $(c, d) R (e, f)$

then $a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2$

Therefore $a^2 + b^2 = e^2 + f^2$

so $(a, b) R (e, f)$

so $R$ is transitive.

So $R$ is an equivalence relation.
6. Number of relations

(a) [5 points] Let $A = \{1\}$. How many different relations on $A$ are there?

\[ 2 \quad \left( \emptyset \quad \text{and} \quad \{(1,1)\} \right). \]

(b) [5 points] Let $A = \{1, 2\}$. How many different relations on $A$ are there?

\[ 2^4 = 16. \]

There are 4 possible pairs:

\[ (1,1), (1,2), (2,1), (2,2) \]

and each one could be in a relation or not.

(c) [5 points] Let $A = \{1, 2, 3\ldots, n\}$. How many different relations on $A$ are there?

\[ 2^{n^2}. \]

There are $n^2$ pairs from $A$.

Each pair can be in the relation or not.
7. A special type of door lock has a panel with five buttons labeled with the
digits 1 through 5. This lock is opened by a sequence of three actions.
Each action consists of either pressing one of the buttons or pressing a pair
of them simultaneously. For example, 12-4-3 is a possible combination.
The combination 12-4-3 is the same as 21-4-3 because both the 12 and
the 21 simply mean to press buttons 1 and 2 simultaneously.

(a) [5 points] How many combinations are possible?

Each step has 15 possibilities (the 5 single digit possibilities) plus the \( \binom{5}{2} = 10 \) 2-digit possibilities

So

\[ 15 \times 15 \times 15 = \sqrt[3]{15^3} \]


(b) [5 points] How many combinations are possible if no digit is repeated
in the combination?

There are 3 possibilities:
1) 2 2-digit combos and 1 1-digit

\[ \binom{5}{2} \binom{3}{1} \binom{1}{1} = 90 \text{ ways} \]

(multiply by 3 to select which step(s) the 1-digit combo

2) 1 2-digit combo and 2 1-digits

\[ \binom{5}{2} \binom{3}{2} \binom{1}{1} = 180 \text{ ways} \]

(multiply by 3 to select the 2-digit combo

3) \( \binom{5}{1} \binom{4}{1} \binom{3}{1} = 60 \text{ ways} \)

Total # of possibilities is \( 90 + 180 + 60 = \sqrt{330} \)