Math 230 Midterm #3
April 4, 2014

Instructions: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

• You may NOT use a calculator.

• Show all of your work.

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1. True or False:
   For the following suppose that $A$, $B$ and $C$ are sets.
   
   (a) [2 points] Every function is a relation.
   
   (b) [2 points] Every relation is a function.
   
   (c) [2 points] The Pigeonhole principle can be stated as: “Let $A$ and $B$ be finite sets and let $f : A \to B$. If $|A| > |B|$, then $f$ is not one-to-one.”
   
   (d) [2 points] The function $f = \{(1, 1), (2, 1), (3, 4)\}$ is a function $f : \{1, 2, 3, 4\} \to \{1, 4\}$.
   
   (e) [2 points] $f(x) = 7x - 12$ is a bijection from $(0, 1)$ to $(-12, -5)$.
   
   (f) [2 points] If $f : A \to B$ is one-to-one and $g : B \to C$ is one-to-one, then $g \circ f$ is one-to-one.
   
   (g) [2 points] Let $f : A \to B$. If $f$ is one-to-one, then $f^{-1}$ is a function and $f^{-1} : B \to A$.
   
   (h) [2 points] If $f : A \to B$ is a bijection, then $f \circ f^{-1} = id_A$.
   
   (i) [2 points] If $f = id_A$, $g = id_B$ such that $A \subseteq B$ and $A \neq B$, then $f \circ g$ is undefined.
   
   (j) [2 points] If there are 13 people in a room, then at least two of them were born on the same month (not necessarily on the same year).
2. For each of the following statements, write the first sentences of a proof by contradiction:

(a) [2 points] If a square of a rational number is an integer, then the rational number must also be an integer.

(b) [2 points] Distinct circles intersect in at most two points.

(c) [2 points] If the sum of two primes is prime, then one of the primes must be 2.

(d) [2 points] There are infinitely many primes.

(e) [2 points] Consecutive integers cannot both be even.
3. Prove the following statements:

(a) [5 points] If the sum of two primes is prime, then one of the primes must be two.

(b) [5 points] Let $A$ and $B$ be sets such that $A \cap B = \emptyset$. Then $(A \times B) \cap (B \times A) = \emptyset$. 
4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = |x|$ and let $g : \mathbb{N} \to \mathbb{N}$ be defined by $g(x) = |x|$.

(a) [5 points] Prove or disprove: $f$ is one-to-one.

(b) [5 points] Prove or disprove: $f$ is onto.
(c) [5 points] Prove or disprove: $g$ is one-to-one.

(d) [5 points] Prove or disprove: $g$ is onto.
5. (a) [5 points] Prove that if \( n \geq 10^{10} \) is a positive integer, then two of its digits must be the same.

(b) [5 points] The squares of an \( 8 \times 8 \) chess board are colored black or white (not necessarily the same way a usual chess board is colored). We call a group of squares an L-region if it consists of a corner square, the two squares above it and the two squares to its right (so it has the shape of an L with equal width and height). Prove that no matter how we color the chess board, there must be two L-regions that are colored identically.
6. [10 points] Let \( A = \{1, 2, 3, 4, 5\} \) with \( f : A \to A \), \( g : A \to A \), and \( h : A \to A \). We are given the following:

- \( f = \{(1, 2), (2, 3), (3, 1), (4, 3), (5, 5)\} \),
- \( h = \{(1, 3), (2, 3), (3, 2), (4, 5), (5, 3)\} \), and
- \( h = f \circ g \).

Find all possible functions \( g \) that satisfy these conditions.
7. [10 points] Let $A, B$ and $C$ be sets. Prove that if $f : A \rightarrow B$ and $g : B \rightarrow C$ are bijections, then $g \circ f$ is a bijection.