Cardinality Homework

April 15, 2014

For the following problems assume that if A is a set, $|A|$ is the cardinality of A.

Problem 1. *In the following problems, find a bijection from A to B (you need not prove that the function you list is a bijection):*

(a) $A = (-3, 3)$, $B = (7, 12)$.

(b) $A = (0, 2)$, $B = (0, 1)$.

(c) $A = (1, 7)$, $B = (-2, 2)$.

(d) $A = \mathbb{N}$, $B = \mathbb{Z}$.

(e) $A = \mathbb{R}$, $B = (0, \infty)$.

(f) $A = \mathbb{N}$, $B = \{\frac{\sqrt{2}}{n} : n \in \mathbb{N}\}$.

(g) $A = \{0, 1\} \times \mathbb{N}$, $B = \mathbb{N}$.

(h) $A = [0, 1]$, $B = (0, 1)$.

Problem 2. *Prove or disprove that the following sets are countable:*

(a) $\{\log n : n \in \mathbb{N}\}$.

(b) $\{(m, n) \in \mathbb{N} \times \mathbb{N} : m \leq n\}$.

(c) $\mathbb{Q}^{100}$.

(d) The set of irrational numbers.

Problem 3. *Let A and B be sets. Prove that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Remark 1. *This result is known as the Cantor-Bernstein-Schöeder Theorem.*

Problem 4. *Prove that $|(0, 1)| = |[0, 1]|$.*

Problem 5. *Dedekind decided he wanted to write a definition of an infinite set that did not depend on the natural numbers. He defined it as follows: "A is an infinite set if there exists a proper subset B of A (that is, $B \subseteq A$ and $A \neq B$) such that $|A| = |B|$." We'll call sets satisfying this condition "Dedekind-infinite" sets.*

(a) Prove that if A is a finite set, then A is not Dedekind-infinite.

(b) Prove that if A is an infinite set, then A is Dedekind-infinite.

Note that proving (a) and (b) means that the natural definition of an infinite set (saying that it is not finite) is equivalent to the Dedekind definition.

Problem 6. *Let $\mathfrak{F}$ be the set of all functions $\mathbb{N} \to \{0, 1\}$. Show that $|\mathbb{R}| = |\mathfrak{F}|$.*

Problem 7. *Let $\mathfrak{F}$ be the set of all functions $\mathbb{R} \to \{0, 1\}$. Show that $|\mathbb{R}| < |\mathfrak{F}|$.*