Cardinality Homework Solutions

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Problem 1. *In the following problems, find a bijection from A to B (you need not prove that the function you list is a bijection):*

(a) $A = (-3, 3)$, $B = (7, 12)$.

(b) $A = (0, 2)$, $B = (0, 1)$.

(c) $A = (1, 7)$, $B = (-2, 2)$.

(d) $A = \mathbb{N}$, $B = \mathbb{Z}$.

(e) $A = \mathbb{R}$, $B = (0, \infty)$.

(f) $A = \mathbb{N}$, $B = \{\frac{\sqrt{2}}{n} : n \in \mathbb{N}\}$.

(g) $A = \{0, 1\} \times \mathbb{N}$, $B = \mathbb{N}$.

(h) $A = [0, 1]$, $B = (0, 1)$.

Solution 1. (a) $f(x) = \dfrac{5}{6}x + \dfrac{57}{6}$.

(b) $f(x) = \dfrac{1}{2}x$.

(c) $f(x) = \dfrac{2}{3}x - \dfrac{8}{3}$.

(d) *The idea is to have $f(1) = 0$ then alternate $f(2) = 1$, $f(3) = -1$, $f(4) = 2$, $f(5) = -2$ and so on. In a pretty math formula it looks as follows:*

$$f(n) = \begin{cases} \dfrac{n}{2} & , \quad \text{if } n \text{ is even,} \\ -\dfrac{n-1}{2} & , \quad \text{if } n \text{ is odd.} \end{cases}$$

(e) $f(x) = e^x$.

(f) $f(x) = \dfrac{\sqrt{2}}{n}$.

(g) *Let $(a, b) \in \{0, 1\} \times \mathbb{N}$. The idea is to send the terms with $a = 0$ to the even integers and the $a = 1$ terms to the odd positive integers. In a pretty formula it looks as follows:*

$$f((a, b)) = \begin{cases} 2b & , \quad \text{if } a = 0, \\ 2b - 1 & , \quad \text{if } a = 1. \end{cases}$$

(h) We discussed in class that the strategy for this was to find a countable subset and play around with it (in this case "deleting" two terms from it). A possible bijection is the following:

$$f(x) = \begin{cases} \dfrac{1}{2} & , \quad \text{if } x = 0, \\ \dfrac{1}{n+2} & , \quad \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ x & , \quad \text{otherwise.} \end{cases}$$

Problem 2. *Prove or disprove that the following sets are countable:*

(a) $A = \{\log n \, : \, n \in \mathbb{N}\}$.

(b) $A = \{(m, n) \in \mathbb{N} \times \mathbb{N} \, : \, m \leq n\}$.

(c) $A = \mathbb{Q}^{100}$.

(d) The set of irrational numbers.

Solution 2. *(a) Let* $f : \mathbb{N} \to A$ *be defined as* $f(n) = \log n$. f *is one-to-one because* $\log x = \log y$ *implies* $x = y$. f *is onto because the image of* f *is exactly the set* A. *Therefore* f *is a bijection. Therefore* A *is countable.*

(b) $A \subseteq \mathbb{N} \times \mathbb{N}$. *Therefore* $|A| \leq |\mathbb{N}|$. *But since* A *is also infinite (indeed,* n *is unbounded in the set), then* $|A| = \mathbb{N}$. *Therefore* A *is countable.*

(c) $\mathbb{Q}^{100} = \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$ *is a cross product of countable sets. Since the finite cross product of countable sets is countable. The set is countable.*

(d) Let A *be the set of irrationals. Since every real number is either rational or irrational, then* $A \cup \mathbb{Q} = \mathbb{R}$. *Suppose for the sake of contradiction that* A *is countable. Then since* $\mathbb{Q}$ *is countable,* $A \cup \mathbb{Q}$ *is also countable. Therefore* $\mathbb{R}$ *is countable. This contradicts the fact that the set of real numbers is uncountable. Therefore* A *is uncountable.*

Problem 3. *Let* A *and* B *be sets. Prove that if* $|A| \leq |B|$ *and* $|B| \leq |A|$, *then* $|A| = |B|$.

Remark 1. *This result is known as the Cantor-Bernstein-Schöeder Theorem.*

Solution 3. *Because* $|A| \leq |B|$, *there exists a one-to-one function* $f : A \to B$. *Because* $|B| \leq |A|$, *there exists a one-to-one function* $g : B \to A$ *and hence there is a bijection* $g^{-1} : Im(g) \to B$. *Note that* g^{-1} *is already a bijection, so consider the following function:*

$$h_1(a) = \begin{cases} g^{-1}(a) & , \quad \text{if } a \in Im(g), \\ f(a) & , \quad \text{if } a \in A - Im(g). \end{cases}$$

Because g^{-1} *is onto,* h_1 *is onto, but it's not one-to-one because any element in the image of* f *is also in the image of* g^{-1}. *So we have an onto function but it't not one-to-one yet. It does however only fail one-to-one-ness in a subset of the image of* f. *We can try to shrink the parts where it fails one-to-one-ness. Consider the following function:*

$$h_2(a) = \begin{cases} g^{-1}(a) & , \quad \text{if } a \in (Im(g) - Im(g \circ f)) \cup (Im(g \circ f \circ g)), \\ f(a) & , \quad \text{if } a \in (A - Im(g)) \cup (Im(g \circ f) - Im(g \circ f \circ g)). \end{cases}$$

h_2 *is onto by construction (using that* g^{-1} *is a bijection) and now the space where* h_2 *fails to be one-to-one is in a subset of the image of* $f \circ g \circ f$ *which is smaller than the image of* f. *We can keep iterating this process and shrink the area that fails the one-to-one-ness while keeping the onto-ness. The question is whether this shrinking leads to the empty set eventually. Note that in* h_1 *we use* $f(a)$ *whenever* $a \in A - Im(g)$ *and in* h_2

we use $f(a)$ whenever $a \in (A - Im(g)) \cup (Im(g \circ f) - Im(g \circ f \circ g)) = (A - Im(g)) \cup ((g \circ f)(A - Im(g)))$. There's a pattern here that can be extended. Let

$$\begin{aligned} G_0 &= A - Im(g), \\ G_1 &= \{g \circ f(a) \,|\, a \in G_0\}, \\ G_n &= \{g \circ f(a) \,|\, a \in G_{n-1}\} \text{ for } n \geq 2. \end{aligned}$$

Now let

$$G = \bigcup_{i=0}^{\infty} G_i.$$

Since every G_i is a subset of A, then $G \subseteq A$. Now define the following function:

$$h(a) = \begin{cases} f(a) & , \quad \text{if } a \in G, \\ g^{-1}(a) & , \quad \text{if } a \notin G. \end{cases}$$

h is a function from A to B because h has a unique output for each input $a \in A$ and because the image of h is a subset of B (since f and g^{-1} are functions to B).

Now let's prove that h is a bijection. First we'll show that h is one-to-one. Suppose $h(x) = h(y)$ for $x, y \in A$. We have three cases:

(i) If x and y are in G, then $h(x) = h(y)$ implies $f(x) = f(y)$ and since f is one-to-one it implies that $x = y$.

(ii) If x and y are not in G, then $h(x) = h(y)$ implies $g^{-1}(x) = g^{-1}(y)$ and since g^{-1} is one-to-one then $x = y$.

(iii) If x is in G and y is not in G, then $h(x) = h(y)$ implies $f(x) = g^{-1}(y)$, therefore $g \circ f(x) = y \notin G$. But since $x \in G$ that means there exists an integer k such that $x \in G_k$. Since $x \in G_k$, then $g \circ f(x) \in G_{k+1}$, so $g \circ f(x) \in G$. This contradicts that $y \notin G$.

Note that we don't need the fourth case $x \notin G$, $y \in G$ because it is analogous to the process in case (iii). Since $h(x) = h(y)$ implies $x = y$, then h is one-to-one.

To finish the theorem we need only prove h is onto. Let $y \in B$. We'll prove it by considering two cases

(i) If $g(y) \notin G$, then $h(g(y)) = g^{-1}(g(y)) = y$, therefore $y \in Im(h)$.

(ii) If $g(y) \in G$, then there exists an integer k such that $g(y) \in G_k$. Since $g(y) \notin A - Im(g)$ (because $g(y)$ must be inside the image of g), then $k \geq 1$. Since $k \geq 1$ and $g(y) \in G_k$, then $g(y) = g \circ f(x)$ for some element $x \in G_{k-1}$. But then $x \in G$, so $h(x) = f(x) = g^{-1}(g(y)) = y$ and hence $y \in Im(h)$.

Therefore h is onto and hence we have a bijection from A to B proving that $|A| = |B|$.

Problem 4. *Prove that $|(0,1)| = |[0,1]|$.*

Solution 4. *We can prove it by showing that the function in exercise (g) of Problem 1 is a bijection but we can prove it quickly by using the Cantor-Bernstein-Schöeder theorem as follows:*

(1) $|(0,1)| \leq |[0,1]|$ because $(0,1) \subseteq [0,1]$. Since $|(0,1)| = |\mathbb{R}|$ this means $|\mathbb{R}| \leq |[0,1]|$.

(2) $|[0,1]| \leq |\mathbb{R}|$ because $[0,1] \subseteq \mathbb{R}$.

Combining (1) and (2) using the Cantor-Bernstein-Schöeder theorem we conclude that $|[0,1]| = |\mathbb{R}| = |(0,1)|$.

Problem 5. *Dedekind decided he wanted to write a definition of an infinite set that did not depend on the natural numbers. He defined it as follows: "A is an infinite set if there exists a proper subset B of A (that is, $B \subseteq A$ and $A \neq B$) such that $|A| = |B|$." We'll call sets satisfying this condition "Dedekind-infinite" sets.*

(a) *Prove that if A is a finite set, then A is not Dedekind-infinite.*

(b) *Prove that if A is an infinite set, then A is Dedekind-infinite.*

Note that proving (a) and (b) means that the natural definition of an infinite set (saying that it is not finite) is equivalent to the Dedekind definition.

Solution 5. *For part (a) suppose for the sake of contradiction that A is finite and A is Dedekind infinite, i.e., there exists a proper subset B of A such that $|A| = |B|$. Since A is finite let $|A| = |B| = n$ for some integer n. Since B is a proper subset of A, then there exists an element $a \in A - B$. $A - B$ and B are disjoint and finite, therefore*

$$n = |A| = |(A - B) \cup B| = |(A - B)| + |B| \geq 1 + n.$$

$n \geq n + 1$ is a contradiction, therefore A is not Dedekind infinite.

Part b is harder. Let A be an infinite set. The goal is to find a proper subset B of A such that $|A| = |B|$. The idea is to let B be A minus one element. Let's figure out how to take one element of A without losing cardinality. Since A is infinite, then we can find a countable subset of A as follows:

- *Since A is non-empty, there exists an $a_1 \in A$. Let $A_1 = A - \{a_1\}$, since A is infinite, A_1 is also infinite.*
- *Since A_1 is non-empty, there exists an $a_2 \in A_1$. Let $A_2 = A - \{a_1, a_2\}$. Since A is infinite, A_2 is also infinite.*
- *Let $n \geq 2$. Since A_{n-1} is non-empty, there exists an $a_n \in A_{n-1}$. Let $A_n = A - \{a_1, a_2, \ldots, a_{n-1}, a_n\}$. Since A is infinite and $\{a_1, \ldots, a_{n-1}, a_n\}$ is finite, then A_n is infinite.*

Following that process we can find an infinite subset $\{a_1, a_2, a_3, \ldots\}$ of A. Now define the following function $f : A \to A_1$:

$$f(x) = \begin{cases} a_{n+1} & , \quad \text{if } x = a_n \text{ for some } n \in \mathbb{N}, \\ x & , \quad \text{otherwise.} \end{cases}$$

The function is one-to-one because if $f(x) = f(y)$ then one of three cases occurs:

- *If $x = a_n$ and $y = a_m$ for some n and m, then $f(x) = f(y)$ implies $a_{n+1} = a_{m+1}$, so $n = m$, so $a_n = a_m$, and hence $x = y$.*
- *If $x = a_n$ for some n and $y \neq a_m$ for any m, then $f(x) = f(y)$ implies $a_{n+1} = y$, but that contradicts that y is not of the form a_m.*
- *If $x \neq a_n$ for any n and $y \neq a_m$ for any m, then $f(x) = f(y)$ implies $x = y$.*

Therefore f is one-to-one. f is onto because if $y \in A_1$, then either $y = a_n$ for some integer $n \geq 2$ and hence $f(a_{n-1}) = a_n = y$ or $y \neq a_n$ for any integer $n \geq 2$ and then $f(y) = y$ unless $y = a_1$. Since $y \in A_1$, then $y \neq a_1$. Therefore for any $y \in A_1$, there exists an $x \in A$ such that $f(x) = y$. This proves f is onto. Since f is onto and one-to-one, $|A| = |A_1|$. Since $A_1 \subseteq A$ and $A_1 \neq A$, then A is Dedekind infinite.

Problem 6. *Let $\mathfrak{F}$ be the set of all functions $\mathbb{N} \to \{0, 1\}$. Show that $|\mathbb{R}| = |\mathfrak{F}|$.*

Solution 6. *We'll prove that $|\mathfrak{F}| = |2^{\mathbb{N}}|$ and since $|2^{\mathbb{N}}| = |\mathbb{R}|$, then $|\mathfrak{F}| = |\mathbb{R}|$. To prove that $|\mathfrak{F}| = |2^{\mathbb{N}}|$, we'll build a bijection $h : \mathfrak{F} \to 2^{\mathbb{N}}$ as follows. For a function $f \in \mathfrak{F}$, i.e., $f : \mathbb{N} \to \{0, 1\}$, define $h(\mathfrak{F})$ to be $\{n \in \mathbb{N} \mid f(n) = 1\}$ i.e., the set of natural numbers that satisfy that $f(n) = 1$.*

First let's prove that h is onto. Suppose $A \in 2^{\mathbb{N}}$, then $A \subseteq \mathbb{N}$. Let f be defined as follows:

$$f(n) = \begin{cases} 1 & , \quad \text{if } n \in A, \\ 0 & , \quad \text{if } n \notin A. \end{cases}$$

Then $h(f) = \{n \in \mathbb{N} \mid f(n) = 1\} = A$. Therefore h is onto.

Now let's prove that h is one-to-one. Suppose $h(f) = h(g)$. Let's prove that $f = g$. Since $f : \mathbb{N} \to \{0, 1\}$ and $g : \mathbb{N} \to \{0, 1\}$, then $dom(f) = dom(g) = \mathbb{N}$, so to show $f = g$ we need only show $f(n) = g(n)$ for all $n \in \mathbb{N}$. Note that there are two cases:

- *If* $f(n) = 1$, *then* $n \in h(f)$, *but since* $h(f) = h(g)$, *then* $n \in h(g)$, *therefore* $g(n) = 1$. *Therefore* $f(n) = g(n) = 1$.
- *If* $f(n) = 0$, *then* $n \notin h(f)$, *but since* $h(f) = g(f)$, *then* $n \notin h(g)$, *therefore* $g(n) = 0$. *Therefore* $f(n) = g(n) = 0$.

In both cases $f(n) = g(n)$, *which allows us to conclude that* $f = g$ *and hence* h *is one-to-one.*

Problem 7. *Let* $\mathfrak{F}$ *be the set of all functions* $\mathbb{R} \to \{0, 1\}$. *Show that* $|\mathbb{R}| < |\mathfrak{F}|$.

Solution 7. *Let* $h : \mathfrak{F} \to 2^{\mathbb{R}}$ *be defined by* $h(f) = \{x \in \mathbb{R} \mid f(x) = 1\}$. *Just like in the previous problem, this function is a bijection. Therefore* $|\mathfrak{F}| = |2^{\mathbb{R}}|$. *By Cantor's theorem we know* $|\mathbb{R}| < |2^{\mathbb{R}}| = |\mathfrak{F}|$. *Which is what we wanted to prove.*