# Homework 1 Solutions

$\boxed{3.1}$ Determine which are true and which are false:

a) $3 \mid 100$. False because $100 = 3\left(\frac{100}{3}\right)$ and $\frac{100}{3}$ is not an integer.

b) $3 \mid 99$. True because $99 = 3 \cdot 33$.

c) $-3 \mid 3$. True because $3 = (-3)(-1)$.

d) $-5 \mid -5$. True because $-5 = (-5)(1)$.

e) $-2 \mid -7$. False because $-7 = (-2)(7/2)$ and $7/2$ is not an integer.

f) $0 \mid 4$. False because $4 \neq 0 \cdot k$ for any integer $k$.

g) $4 \mid 0$. True because $0 = 4 \cdot 0$ and $0$ is an integer.

h) $0 \mid 0$. True because $0 = 0 \cdot 1$ and $1$ is an integer.

$\boxed{3.4}$ $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$

Let's define $\leq$ using $\mathbb{N}_0$.

If $a, b \in \mathbb{Z}$ and $a - b \in \mathbb{N}_0$ then $b \leq a$ and $a \geq b$.

If $a, b \in \mathbb{Z}$ and $a - b \in \mathbb{N}_0$ and $a \neq b$ then $b < a$ and $a > b$.

$\boxed{3.5}$ Every integer is rational because if $n$ is an integer then $n = \frac{n}{1}$ so it is a ratio of two integers.

Not all rational numbers are integers because $\frac{1}{2}$ is not an integer yet it is rational.

$\boxed{3.7}$ $n$ is the squareroot of a number $m$ if $n^2 = m$.

$\boxed{3.12}$ How many positive divisors do each of the following have?

a) 8 : 4 divisors

b) 32 : 6 divisors

c) $2^n$ : $n+1$ divisors

d) 10 : 4 divisors

e) 100 : 9 divisors

f) 1 000 000 : 49 divisors

g) $10^n$ : $(n+1)^2$ divisors

h) $30 = 2 \times 3 \times 5$ : 8 divisors

i) $42 = 2 \times 3 \times 7$ : 8 divisors. Same as 30 because they have the same number of prime factors raised to the same power.

j) $2310 = 2 \times 3 \times 5 \times 7 \times 11$ : 32 divisors.

k) $1\times 2\times 3\times 4\times 5\times 6\times 7\times 8 = 2^7\times 3^2\times 5\times 7$

The divisors are numbers of the form $2^{\alpha_1}\cdot 3^{\alpha_2}\cdot 5^{\alpha_3}\cdot 7^{\alpha_4}$

where $0\le\alpha_1\le 7$, $0\le\alpha_2\le 2$, $0\le\alpha_3\le 1$ and $0\le\alpha_4\le 1$

so there are $8\times 3\times 2\times 2 =$ 96 divisors of $8!$

l) 0. $d|0$ if $\exists$ an integer $a$ s.t $0=d\cdot a$
Since $a=0$ works for any $d$.

0 has infinitely many divisors.

3.13 a) 6
b) The next perfect number is 496

4.1 Rewrite in the form "If A, then B"

a) If $x$ and $y$ are even and odd integers, respectively, then $xy$ is even.

b) If $x$ is a odd, then $x^2$ is odd.

c) If $p$ is prime, then $p^2$ is not prime.

d) If $a$ and $b$ are negative numbers, then $ab$ is negative.

e) If $ABCD$ is a rhombus with diagonal $\overline{AC}$ and $\overline{BD}$, then $\overline{AC}$ is perpendicular to $\overline{BD}$.

f) If $\Delta ABC$ and $\Delta DEF$ are congruent triangles, then the area of $\Delta ABC$ equals the area of $\Delta DEF$.

g) If $a$, $b$ and $c$ are consecutive integers, then $3 \mid a+b+c$.

$\boxed{4.2}$ I will write $\rightarrow$ if "If A, then B" is true, $\leftarrow$ if "If B, then A" and $\longleftrightarrow$ if "A if and only if B".

a) $\leftarrow$ (Every square is a rectangle but not every rectangle is a square)

b) $\longrightarrow$ (Every rectangle is a parallellogram but not every parallellogram is a rectangle)

c) $\longrightarrow$ (Being a grandfather implies one is male. Being male does not imply one is a grandfather)

d) $\longrightarrow$

e) $\leftarrow$ (Every leap year is a multiple of 4, but not all multiples of 4 are leap years, for example 1900, 1800, 1700 were not leap years and 2100, 2200, 2300 won't be leap years)

f) Neither.

g) $\longrightarrow$

h) $\longleftrightarrow$

i) $\longleftrightarrow$

j) $\leftarrow$

k) $\leftarrow$

$\boxed{4.7}$ Because the equilateral triangle is not a right triangle.

# Homework #2
## SOLUTIONS

**5.1** Proof:
Let x and y be odd integers.
Since x and y are odd there exist a and b such that $x=2a+1$ and $y=2b+1$.

$$\begin{aligned} x+y &= 2a+1+2b+1 \\ &= 2(a+b)+2 \\ &= 2(a+b+1) \\ &= 2c \end{aligned}$$

where c is the integer $a+b+1$.
Therefore there exists an integer c such that
$x+y=2c$,
therefore $x+y$ is even $\boxed{X}$

**5.7** Proof:
We want to show "If n is odd then $n^2$ is odd"

Let n be an odd integer. Therefore $n=2a+1$ for some integer a.

$$\begin{aligned} n^2=(2a+1)^2 &= 4a^2+4a+1 \\ &= 2(2a^2+2a)+1 \\ &= 2c+1 \end{aligned}$$

where $c=2a^2+2a$ is an integer.
Therefore $n^2=2c+1$ for some c, so $n^2$ is an odd integer. $\boxed{X}$

**5.11** Proof: Let a, b, d, x and y be integers.

Since $d|a$ there exists m such that $a=dm$.
Since $d|b$ there exists n such that $b=dn$.

Therefore $ax+by = dmx+dny = d(mx+ny) = dc$, where $c$ is the integer $mx+ny$.

Therefore $ax+by=dc$ with $c$ an integer
Therefore $d \mid ax+by$.

5.18 Proof: Let $m$ and $n$ be consecutive perfect squares with $n>m$.
Since $m$ and $n$ are ~~a~~ perfect squares then $m=a^2$ and $n=b^2$ for some integer $a$ and $b$.
Since $n>m$, $b^2>a^2$. Since they are consecutive squares $b=a+1$.
Therefore
$$\begin{aligned} n-m &= b^2-a^2 \\ &= (a+1)^2-a^2 \\ &= a^2+2a+1-a^2 \\ &= 2a+1 \end{aligned}$$
Therefore $n-m$ is odd.

(Note I assumed $n>m$, but that is not given, so technically one must do both cases $n>m$ and $m>n$, I avoid that here since both cases are symmetric).

5.23 If you only prove "If A, then C" then it could be the case that for some true instances of B, C is false, so then you'd have ~~an~~ true instance of A or B where C is false, hence "If A or B, then C" is false in that case.
Therefore you need to prove both "If A, then C" and "If B, then C". Now the reason proving both proves ~~everyt~~ "If A or B, then C" is that A or B means either an instance of A is true or an instance of B is true, but we know that in either case, C is forced to be true showing "If A or B, then C".

6.1 $a=1$, $b=-1$ is a counterexample.
Indeed $+1 \mid -1$ and $+1 \leq -1$ is false.

6.3 $a=6$, $b=2$, $c=3$ is a counter example.
$6 \mid (2)(3)$ but $6 \nmid 2$ and $6 \nmid 3$.

6.6 $p=11$ is a counterexample because $2^{11}-1$ is composite

6.9 a) List them
b) $n=41$ is a counterexample because
$41^2+41+41$ is a multiple of 41, hence composite
(note $n=40$ is also a counterexample).

6.13 $n=30$ has 3 prime factors and it is composite,
so it's a counterexample.

7.1 a) F  b) T  c) F  d) F  e) T

7.6

| $x$ | $y$ | $x \Leftrightarrow y$ | $x \Rightarrow y$ | $y \Rightarrow x$ | $(x \Rightarrow y) \wedge (y \Rightarrow x)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | F | T | F |
| F | T | F | T | F | F |
| F | F | T | T | T | T |

Since the columns for $x \Leftrightarrow y$ and $(x \Rightarrow y) \wedge (y \Rightarrow x)$ are the same they are equivalent statements. ☒

7.10

| $x$ | $y$ | $x \to y$ | $y \to x$ | $x \leftrightarrow y$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

a) When $x = T$ and $y = F$ $x \to y$ is T but $y \to x$ is F.

b) When $x = F$ and $y = T$ $x \to y$ is T but $x \leftrightarrow y$ is F.

c) If $x = T$ and $y = T$ then $x \vee y = T$

but $(x \wedge \neg y) \vee ((\neg x) \wedge y) = (T \wedge (\neg T)) \vee ((\neg T) \wedge T)$

$= (T \wedge F) \vee (T \wedge F)$

$= F \vee F = F$

So they can't be logically equivalent.

7.11 a)

| $x$ | $y$ | $(x \vee y) \vee (x \vee \neg y)$ |
|---|---|---|
| T | T | $(T \vee T) \vee (T \vee F) = T \vee T = T$ |
| T | F | $(T \vee F) \vee (T \vee T) = T \vee T = T$ |
| F | T | $(F \vee T) \vee (F \vee F) = T \vee F = T$ |
| F | F | $(F \vee F) \vee (F \vee T) = F \vee T = T$ |

c)

| $x$ | $y$ | $(\neg(\neg x)) \leftrightarrow x$ |
|---|---|---|
| T | T | $(\neg F) \leftrightarrow T = T \leftrightarrow T = T$ |
| T | F | $(\neg F) \leftrightarrow T = T \leftrightarrow T = T$ |
| F | T | $(\neg T) \leftrightarrow F = F \leftrightarrow F = T$ |
| F | F | $(\neg T) \leftrightarrow F = F \leftrightarrow F = T$ |

e)

| $x$ | $y$ | $z$ | $((x \to y) \wedge (y \to z)) \to (x \to z)$ |
|---|---|---|---|
| T | T | T | $(T \to T \wedge (T \to T)) \to (T \to T) = T \to T = T$ |
| T | T | F | $(T \to T \wedge (T \to F)) \to (T \to F) = (T \wedge F) \to F = F \to F = T$ |
| T | F | T | $(T \to F \wedge F \to T) \to (T \to T) = F \to T = T$ |
| T | F | F | $(T \to F \wedge F \to F) \to (T \to F) = F \to F = T$ |
| F | T | T | $(F \to T \wedge T \to T) \to (F \to T) = T \to T = T$ |
| F | T | F | $(F \to T \wedge T \to F) \to (F \to F) = F \to T = T$ |
| F | F | T | $(F \to F \wedge F \to T) \to (F \to T) = T \to T = T$ |
| F | F | F | $(F \to F \wedge (F \to F)) \to (F \to F) = T \wedge T = T$ |

So all cases yield true.

7.13 a) If $x$ is true then $(x \vee y) \wedge (x \vee \neg y) \wedge \neg x = \text{False}$ since $\neg x$ is False.

Assume $x$ is false. Then we're left with cases $y = T$ and $y = F$.
when $y = T$ $(x \vee \neg y) = F$ so $(x \vee y) \wedge (x \vee \neg y) = F$ so the whole thing is false.
when $y = F$ $(x \vee y) = F$ and so everything is false.

(You can also do it with a truth table)

b)

| $x$ | $y$ | $x \wedge (x \Rightarrow y) \wedge (\neg y)$ |
|---|---|---|
| T | T | $T \wedge (T \Rightarrow T) \wedge F = F$ |
| T | F | $T \wedge (T \rightarrow F) \wedge T = F$ |
| F | T | $F \wedge (\quad) \wedge (\,) = F$ |
| F | F | $F \wedge (\quad) \wedge (\,) = F$ |

c)

| $x$ | $y$ | $(x \rightarrow y) \wedge ((\neg x) \rightarrow y) \wedge \neg y$ |
|---|---|---|
| T | T | $T \wedge (F \rightarrow T) \wedge F = F$ |
| T | F | $F \wedge (\quad) \wedge (\,) = F$ |
| F | T | $\wedge (T \rightarrow T) \wedge F = F$ |
| F | F | $T \wedge (T \rightarrow F) \wedge T = F$ |

7.17 There are 16, 2 choices for $(T * T)$, 2 for $(T * F)$
2 for $F * T$ and 2 for $F * F$

so $2^4 = 16$ possibilities.