Homework 7 Solutions
Math 230

25.6: If two numbers match zeroes, then they satisfy that the zeroes of both numbers are in the same positions among the 9 digits of the numbers. There are 512 configurations of 0's and not 0's (indeed each digit is either a 0 or not a 0 and there are 9 digits, hence $2^9 = 512$ configurations). Since we have 513 numbers, by the Pigeonhole principle at least two of them have the same configuration of 0's. Hence they match zeroes.

25.9: Break the square into 4 squares of side length $1/2 \times 1/2$ (i.e. draw the lines connecting the midpoints of opposing sides of the square). Since there are 5 points, at least two of them must land in the same $1/2 \times 1/2$ square. The farthest apart two points can be inside the square is if they are in opposing corners, hence

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$$

apart. This is what we wanted to prove.

25.16: For this exercise we must take $\mathbb{N}$ to be $\{0, 1, 2, \ldots\}$ as opposed to our usual definition we use in class. It's easy to change the function to work out in our case too (add one to $-n/2$ and it still is a bijection while working from our usual definition of $\mathbb{N}$).

Without further ado, let's prove the statement as written in the book. Let's start by proving f is onto:

First note that $f(0) = 0$, hence $0 \in Im(f)$. Now, if $m < 0$ is an integer, then $n = -2m$ is a positive integer, so

$$f(n) = f(-2m) = -\frac{-2m}{2} = m.$$

Therefore $m \in Im(f)$, i.e., all negative integers are in the image of f.

If $m > 0$ is an integer, then $m \geq 1$ so $2m - 1 \geq 1$, so $n = 2m - 1$ is in $\mathbb{N}$ and

$$f(n) = f(2m - 1) = \frac{2m - 1 + 1}{2} = m.$$

Therefore $m \in Im(f)$, i.e., all positive integers are in the image of f.

Therefore all integers m are in the image of f, showing that f is onto $\mathbb{Z}$.

Now, let's prove f is one-to-one:

Suppose $f(n) = f(m)$. Then we have four possibilities:

1. $f(n) = -n/2$ and $f(m) = -m/2$. Then $-n/2 = -m/2$ so $m = n$.

2. $f(n) = -n/2$ and $f(m) = (m+1)/2$. Since n and m are in $\mathbb{N}$, then

$$\frac{m+1}{2} > 0 \geq -\frac{n}{2},$$

so $f(m) > f(n)$, contradicting the assumption that $f(n) = f(m)$.

3. $f(n) = (n+1)/2$ and $f(m) = -m/2$. Since n and m are in $\mathbb{N}$, then

$$\frac{n+1}{2} > 0 \geq -\frac{m}{2},$$

therefore $f(n) > f(m)$ contradicting the assumption $f(n) = f(m)$.

4. $f(n) = (n+1)/2$ and $f(m) = (m+1)/2$. Then $(n+1)/2 = (m+1)/2$, so $n = m$.

Looking at the possibilities we conclude that if $f(n) = f(m)$ then $n = m$ which implies that f is one-to-one.

Since f is onto and f is one-to-one, f is a bijection.

25.18: Proving this would be a generalization of the Pigeonhole Principle to infinite sets (the Pigeonhole principle as stated before was only for finite sets).

Let's prove it by contradiction using Cantor's Theorem. Let A be non-empty and $f : 2^A \to A$. For the sake of contradiction suppose that f is one-to-one. Since f is one-to-one, then f has an inverse $f^{-1} : Im(f) \to 2^A$ that is onto. But the image of f is a subset of A so it's easy to build an onto function $g : A \to 2^A$ by saying $g(a) = f^{-1}(a)$ for $a \in Im(f)$ and $g(a) =$whatevs for $a \notin Im(f)$. Here's an example of a function g that works:

$$g(x) = \begin{cases} f^{-1}(x) & \text{if } x \in Im(f) \\ \emptyset & \text{if } x \notin Im(f) \end{cases}$$

Since f^{-1} is onto and g is an extension of f^{-1}, then $g : A \to 2^A$ is also onto. However, Cantor's theorem states that there are no onto functions from A to 2^A. CONTRADICTION! Hence f is not one-to-one!

Summary of the proof: If f is one-to-one, then f^{-1} is onto. But then there is an onto function from A to 2^A. This is impossible, hence f is not one-to-one.

26.1:
a) $f \circ g = \{(2,2),(3,2),(4,2)\}$ and $g \circ f = \{(1,1),(2,1),(3,1)\}$. $g \circ f \neq f \circ g$.
b) $f \circ g = \{(2,2),(3,3),(4,4)\}$ and $g \circ f = \{(1,1),(2,2),(3,3)\}$. $g \circ f = f \circ g$.
c) $f \circ g$ is undefined. $g \circ f = \{(1,0),(2,5),(3,3)\}$. $g \circ f \neq f \circ g$.
d) $f \circ g = \{(1,4),(2,4),(3,1),(4,1)\}$ and $g \circ f = \{(1,4),(2,4),(3,4),(4,1)\}$. $g \circ f \neq f \circ g$.
e) $f \circ g = \{(1,4),(2,5),(3,1),(4,2),(5,3)\} = g \circ f$. $g \circ f = f \circ g$.
f)

$$f \circ g(x) = f(x^2+1) = (x^2+1)^2 - 1 = x^4 + 2x^2,$$

and

$$g \circ f(x) = g(x^2-1) = (x^2-1)^2 + 1 = x^4 - 2x^2 + 2.$$

$g \circ f(0) \neq f \circ g(0)$, so $g \circ f \neq f \circ g$.
g)

$$f \circ g(x) = f(x-7) = (x-7) + 3 = x - 4,$$

and

$$g \circ f(x) = g(x+3) = (x+3) - 7 = x - 4.$$

Therefore $g \circ f(x) = f \circ g(x)$.

h)

$$f \circ g(x) = f(2-x) = 1 - (2-x) = x - 1,$$

and

$$g \circ f(x) = g(1-x) = 2 - (1-x) = x + 1.$$

$g \circ f(0) \neq f \circ g(0)$, so $g \circ f \neq f \circ g$.

i) $f \circ g$ is undefined because $g(-1) = 0$ so $f(g(-1))$ is undefined.

$$g \circ f(x) = g\left(\frac{1}{x}\right) = \frac{1}{x} + 1.$$

Since $f \circ g$ is undefined, then $g \circ f \neq f \circ g$.

j) Since $A \neq B$ and $A \subseteq B$, there is an $x \in B$ such that $x \notin A$. For this x, $g(x) = id_B(x) = x$, but $f(x) = id_A(x)$ is undefined. Therefore $f \circ g$ is undefined.

$$g \circ f(x) = g(f(x)) = g(id_A(x)) = g(x) = id_B(x) = x.$$

Since $f \circ g$ is undefined, then $g \circ f \neq f \circ g$.

26.7: Let A and B be sets and $f : A \to B$ and $g : B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$. We want to prove that f is invertible, i.e, that f is one-to-one. We also want to prove that $g = f^{-1}$.

Let's start by proving that f is one-to-one. Suppose that $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$, so $g \circ f(x) = g \circ f(y)$, but $g \circ f = id_A$, so $id_A(x) = id_A(y)$, and therefore $x = y$. Hence f is one-to one, which implies that f is invertible.

Let's now prove that $g = f^{-1}$:

We'll start by proving that f is onto. Let $y \in B$. Since $y \in B$, then $g(y) \in A$. Now $f \circ g(y) = id_B(y) = y$ and $f \circ g(y) = f(g(y))$. So $f(g(y)) = y$, so f is onto.

Since f is one-to-one, f^{-1} exists and its domain is the image of f. Since f is onto, the image of f is B, so $f^{-1} : B \to A$. Therefore the domain of f^{-1} equals the domain of g.

Now we just need to prove that for $y \in B$, $f^{-1}(y) = g(y)$. Since f is onto, there exists an $x \in A$ such that $f(x) = y$. Therefore

$$f^{-1}(y) = f^{-1}(f(x)) = f^{-1} \circ f(x) = id_A(x) = x,$$

and

$$g(y) = g(f(x)) = g \circ f(x) = id_A(x) = x.$$

Therefore $f^{-1}(y) = g(y)$. Therefore $g = f^{-1}$.

26.9: Let A, B, C be sets and $f : A \to B$ and $g : B \to C$.

a) Suppose f and g are one-to-one. Let's prove $g \circ f$ is also one-to-one. Suppose $g \circ f(x) = g \circ f(y)$. Then $g(f(x)) = g(f(y))$. Since g is one-to-one then $f(x) = f(y)$. Since f is one-to-one then $x = y$. Hence $g \circ f$ is one-to-one.

b) Suppose f and g are onto. Let's prove $g \circ f$ is onto. Suppose $c \in C$. Since g is onto, there exists a $b \in B$ such that $g(b) = c$. Since f is onto, there exists an $a \in A$ such that $f(a) = b$. Therefore $g(f(a)) = c$. Therefore $g \circ f(a) = c$. Therefore $g \circ f$ is onto.

c) Suppose f and g are bijections. Let's prove $g \circ f$ is a bijection. Since f and g are one-to-one, then $g \circ f$ is one-to-one. Since f and g are onto, then $g \circ f$ is onto. Since $g \circ f$ is one-to-one and onto, then $g \circ f$ is a bijection.

26.10: The functions in (e) of exercise 26.1 work, i.e., $A = \{1, 2, 3, 4, 5\}$ with

$$f = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\},$$

and

$$g = \{(1, 3), (2, 4), (3, 5), (4, 1), (5, 2)\}.$$

Note that g is not the inverse of f, that neither is the identity and that

$$f \circ g = \{(1, 4), (2, 5), (3, 1), (4, 2), (5, 3)\} = g \circ f.$$

You might be wondering "what's so special about these two functions?" Both of them are permutations of the set $\{1, 2, 3, 4, 5\}$, one of them is a translation by 1 and the other by 2. So the composition is translating by 3. This kind of construction can be easily generalized to find many more functions with the property of $f \circ g = g \circ f$. An interesting question is whether we can characterize all of the pairs of functions (f, g) with such a property.