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MATH 230 MIDTERM #1

February 7, 2014

INSTRUCTIONS: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- You may NOT use a calculator.
- Show all of your work.

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	10	
7	20	
8	20	
Total:	150	

1. True or False (Just answer true or false, you don't need to explain your answer).

(a) [2 points] $T \subseteq A$ if and only if $T \in 2^A$.

TRUE, by definition.

(b) [2 points] There is no x such that $x \subseteq \{x\}$.

FALSE, $x = \phi$

(c) [2 points] If x is a real number and $x^2 < 0$, then x is a perfect number.

TRUE (vacuous truth, no real number x satisfies $x^2 < 0$)

(d) [2 points] Two right triangles that have hypotenuses of the same length have the same area.

FALSE (counterexample in page 6).

(e) [2 points] $\exists x, \forall y, xy = 0$.

TRUE, $(x = 0)$

(f) [2 points] $\forall x, \exists y, xy = 0$.

TRUE, $(y = 0)$

(g) [2 points] $\mathbb{N} \in 2^{\mathbb{Z}}$.

TRUE because $\mathbb{N} \subseteq \mathbb{Z}$.

(h) [2 points] $\{2\} \subseteq \{\{1\}, \{2\}, \{3\}\}$.

FALSE $(\{2\} \in \{\{1\}, \{2\}, \{3\}\})$.

(i) [2 points] If A and B are sets then $2^A \subseteq 2^B$.

FALSE, it's true if $A \subseteq B$.

(j) [2 points] A negation of the statement "There is a natural number that is prime and even" can be phrased as "All natural numbers that are prime are odd".

TRUE.

2. For the following pairs of statements A, B, write a if the statement "If A, then B" is true, write b if the statement "If B, then A" is true, write c if the statement "A if and only if B is true", and write d if none of the statements are true. You should write all that apply. Note that in the following, x and y are integers.

   (a) [5 points] A: $xy = 0$. B: $x = 0$ and $y = 0$.

   b

   (b) [5 points] A: Lines l_1 and l_2 are parallel. B: Lines l_1 and l_2 are perpendicular.

   d

   (c) [5 points] A: Joe is a grandfather. B: Joe is male.

   a

   (d) [5 points] A: $x < 0$ B: $x^3 < 0$.

   a, b, c.

3. Proofs:

   (a) [10 points] Let x be an integer. Prove that x is odd if and only if there is an integer b such that $x = 2b - 1$.

($\Rightarrow$) Let x be odd.

Then $\exists\ a \in \mathbb{Z}$ s.t $x = 2a + 1$.

Therefore $x = 2(a+1) - 1$.

Since $a+1 \in \mathbb{Z}$, let $b = a+1$.

Then $x = 2b - 1$, which is what we wanted.

($\Leftarrow$) Let $x = 2b - 1$ for some integer b.

Then $x = 2(b-1) + 1$ so

$x = 2a + 1$ where $a \in \mathbb{Z}$ $(a = b-1)$.

So x is odd

$\boxtimes$

(b) [5 points] For real numbers a and b, prove that if $0 < a < b$, then $a^2 < b^2$.

Since $a < b$ and $a > 0$ by the properties of products in inequalities

$(a)(a) < a(b)$ so $a^2 < ab$.

Since $b > a$ and $b > 0$ then $b^2 > ab$.

So $a^2 < ab$ and $ab < b^2$.

By the transitive property of "$<$"

we conclude that $a^2 < b^2$.

(c) [5 points] Let A, B and C be sets satisfying $A \subseteq B$ and $B \subseteq C$. Prove that $A \subseteq C$.

Let $x \in A$. Since $A \subseteq B \Rightarrow x \in B$.

$x \in B$ and $B \subseteq C$ so $x \in C$.

Therefore $A \subseteq C$.

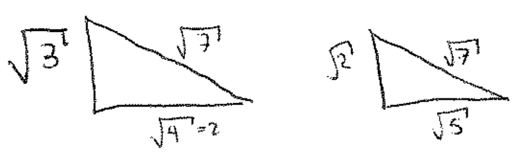
4. Find counterexamples to disprove the following statements:

(a) [5 points] If a, b and c are positive integers with $a|(bc)$, then $a|b$ or $a|c$.

$6 \mid 2 \cdot 3$ but $6 \nmid 2$ and $6 \nmid 3$

so $b = 2$, $c = 3$ and $a = 6$ is a counterexample.

(b) [5 points] Two right triangles have the same area if and only if the lengths of their hypotenuses are the same.



Both have hypotenuse $\sqrt{7}$ but their areas are $\frac{\sqrt{2}\cdot 2}{2} = \sqrt{2}$ and $\frac{\sqrt{2}\cdot\sqrt{5}}{2} = \frac{\sqrt{10}}{2}$

$\sqrt{2} \neq \frac{\sqrt{10}}{2}$ so not all right triangles with the same hypotenuse have the same area.

(c) [5 points] For real numbers a and b, if $a < b$, then $a^2 < b^2$.

Let $a = -2$ and $b = 1$.

$a^2 = 4$ and $b^2 = 1$

so $a^2 > b^2$ while $a < b$.

(d) [5 points] Let A and B be sets. Then $(A \cup B) - B = A$.

If $A = \{1,2\}$, $B = \{2,3\}$ then $(A \cup B) - B = \{1,2,3\} - \{2,3\} = \{1\}$

while

$A = \{1,2\}$

so $(A \cup B) - B \neq A$.

5. Boolean Algebra

(a) [5 points] Prove or disprove the following Boolean expression identity:

$$(x \wedge y) \vee (x \wedge \neg y) = x.$$

· Proof with a truth table:

x	y	$x \wedge y$	$x \wedge (\neg y)$	$(x \wedge y) \vee (x \wedge \neg y)$
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

same.

Since they have the same column they are logically equivalent.

Alternative Proof: Case 1: $x = T$ then $(x \wedge y) \vee (x \wedge \neg y) = (T \wedge y) \vee (T \wedge \neg y)$
$= y \vee (\neg y) = T.$

Case 2: $x = F$ then $(x \wedge y) \vee (x \wedge \neg y) = (F \wedge y) \vee (F \wedge \neg y) = F \vee F = F.$

(b) [5 points] Besides the classic Boolean operations $\wedge, \vee, \neg, \rightarrow, \leftarrow$, we have others, an example of one is the "nand" operation denoted by $\bar{\wedge}$. We define $x \bar{\wedge} y$ to be $\neg(x \wedge y)$. Construct a truth table for $\bar{\wedge}$.

x	y	$x \wedge y$	$\neg(x \wedge y)$	$x \bar{\wedge} y$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

(c) [5 points] Prove or disprove that $\bar{\wedge}$ is commutative.

Let's prove $x \bar{\wedge} y = y \bar{\wedge} x$

x	y	$x \bar{\wedge} y$	$y \bar{\wedge} x$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

They are equal so $\bar{\wedge}$ is commutative.

(d) [5 points] Prove or disprove that $\bar{\wedge}$ is associative.

Let's prove $(x \bar{\wedge} y) \bar{\wedge} z = x \bar{\wedge} (y \bar{\wedge} z)$

x	y	z	$x \bar{\wedge} y$	$(x \bar{\wedge} y) \bar{\wedge} z$	$y \bar{\wedge} z$	$x \bar{\wedge} (y \bar{\wedge} z)$
T	T	T	F	T	F	T
T	T	F	F	T	T	F
T	F	T	T	F	T	F
T	F	F	T	T	T	F
F	T	T	T	F	F	T
F	T	F	T	T	T	T
F	F	T	T	F	T	T
F	F	F	T	T	T	T

Since are different, it is not associative.

6. In my comic book library I have 15 Daredevil paperbacks, 12 Spider-man paperbacks and 3 Batman paperbacks

   (a) [5 points] In how many different ways can these trade paperbacks be arranged on a bookshelf?

$$15+12+3=30.$$

30! ways

(30 factorial)

   (b) [5 points] In how many different ways can these trade paperbacks be arranged on a bookshelf if all the books of the same character are grouped together?

order the characters

$$\boxed{3! \times 15! \times 12! \times 3!}$$

order the Daredevils

Order the spideys

order the batmans

7. Write out the following sets by listing their elements between curly braces.

(a) [5 points] $\{x \in \mathbb{N} : x \leq 10 \text{ and } 3|x\}$.

$\boxed{\{3, 6, 9\}}$

(b) [5 points] $\{x \in \mathbb{Z} : x^2 = 4\}$.

$\boxed{\{-2, 2\}}$

(c) [5 points] $\{x \in \mathbb{Z} : 10|x \text{ and } x|100\}$.

$\boxed{\{10, -10, 20, -20, 50, -50, 100, -100\}}$

(d) [5 points] $\{x : x \subseteq \{1, 2, 3, 4, 5\} \text{ and } |x| \leq 1\}$.

$\{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$

8. Let $A \times B = \{(1,2),(1,3),(1,7),(2,2),(2,3),(2,7),(6,2),(6,3),(6,7)\}$.

(a) [5 points] What is $A \cup B$?

$A = \{1, 2, 6\}$
$B = \{2, 3, 7\}$

$A \cup B = \{1, 2, 3, 6, 7\}$

(b) [5 points] What is $A \cap B$?

$A \cap B = \{2\}$

(c) [5 points] What is $A - B$?

$A - B = \{1, 6\}$

(d) [5 points] What is $A \Delta B$?

$$
\begin{aligned}
A \Delta B &= (A - B) \cup (B - A) \\
&= \{1, 6\} \cup \{3, 7\} \\
&= \{1, 3, 6, 7\}
\end{aligned}
$$