NAME: ______________________

MATH 230 MIDTERM #3

November 25, 2013

INSTRUCTIONS: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- You may NOT use a calculator.
- Show all of your work.

Question	Points	Score
1	20	
2	15	
3	10	
4	20	
5	20	
6	10	
7	15	
Total:	110	

1. True or False (Cardinality Edition, **if you get it wrong you lose 1 point**, so be extra careful):

TRUE (a) [2 points] Two sets have the same cardinality if there exists a bijection from one of them to the other.

By definition

FALSE (b) [2 points] The cardinality of $\mathbb{N}$ is the same as the cardinality of $\mathbb{R}$.

Proved in class that they have different cardinality.

TRUE (c) [2 points] The cardinality of $2^{\mathbb{N}}$ is the same as the cardinality of $\mathbb{R}$.

Proved in class.

TRUE (d) [2 points] The cardinality of $(0,1)$ is the same as the cardinality of $[0,1]$.

$(0,1) \subseteq [0,1]$ so $|(0,1)| \le |[0,1]|$

$[0,1] \subseteq \mathbb{R}$ so $|[0,1]| \le |\mathbb{R}|$ but $|(0,1)| = |\mathbb{R}|$

so $|[0,1]| = |\mathbb{R}| = |(0,1)|$

FALSE (e) [2 points] $f(x) = 2x - 1$ is a bijection from $(0,1)$ to $(0,2)$.

Bijection from $(0,1)$ to $(-1,1)$.

TRUE (f) [2 points] If $f : A \to B$ is onto and $g : B \to C$ is onto, then $g \circ f$ is onto.

(HW exercise)

FALSE (g) [2 points] Suppose $|A| > |B|$, then there is no one-to-one function $f : B \to A$.

(It would be true if it said $f : A \to B$).

TRUE (h) [2 points] Cantor's theorem states that there is no onto function $f : A \to 2^A$.

By definition.

FALSE (i) [2 points] Suppose $f : A \to B$ is one-to-one and $g : B \to A$ is one-to-one. Then f is onto.

Not necessarily. Example: $A = \mathbb{Z}$, $B = \mathbb{Z}$ ~~[illegible]~~

Then $f : A \to B$ can be $f(x) = 2x$

$g : B \to A$ can be $g(x) = 2x$.

Both are one-to-one. Neither is onto

FALSE (j) [2 points] $f(x) = \tan x$ is a bijection from $(-1,1)$ to $\mathbb{R}$.

It's a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$.

2. Determine if the following sets are functions and **explain** why or why not:

   (a) [5 points] $f = \{(x, y) \mid x + y = 0\}$.

Yes.

For each x there is a unique y, namely $y=-x$.

   (b) [5 points] $f = \{(x, y) \mid xy = 0\}$.

No.

If $x=0$ then there are infinitely many y ~~that~~ that satisfy $xy=0$.

note: If $x \in \mathbb{N}$ then $f: \mathbb{N} \to \{0\}$ is a function. So for it not to be a function we assume the set we're dealing with is $\mathbb{R}$.

   (c) [5 points] $f = \{(x, y) \mid x \text{ divides } y\}$.

Not a function.

$1|1$

$1|2$

$1|3$

$1|4$ and so on so if $x=1$, there are many choices of y.

3. Let f be a function. f^{-1} is usually reserved to symbolize the inverse of a function, let's extend this definition in the following way: Suppose $f : A \to B$ is a function and Y is a subset of B. Then we define $f^{-1}(Y)$ to be the set of all elements in A that are mapped to a value in Y. That is,

$$f^{-1}(Y) = \{x \in A : f(x) \in Y\}.$$

For example, suppose $f : \mathbb{Z} \to \mathbb{Z}$ with f defined as $f(x) = x^2$. Then $f^{-1}(\{1,2,3,4\}) = \{1,-1,2,-2\}$. (Note that $\sqrt{2}$, $-\sqrt{2}$, $\sqrt{3}$ and $-\sqrt{3}$ are not integers which is why those numbers don't appear in $f^{-1}(\{1,2,3,4\})$.)

(a) [5 points] Let $f : \mathbb{Z} \to \mathbb{Z}$ be $f(x) = |x|$ (absolute value). If $Y = \{1,2,3\}$, what is $f^{-1}(Y)$.

$$f^{-1}(\{1,2,3\}) = \boxed{\{1,-1,2,-2,3,-3\}}.$$

(b) [5 points] Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$. If $Y = [1,2]$, what is $f^{-1}(Y)$.

$$f^{-1}(Y) = \boxed{[1,\sqrt{2}] \cup [-\sqrt{2},-1]}.$$

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^3$ and let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^3$.

   (a) [5 points] Prove or disprove: f is one-to-one.

Suppose $f(x)=f(y)$ so $x^3 = y^3$

Then $x^3 - y^3 = 0$

$$(x-y)(x^2+xy+y^2)=0$$

Either $x=y$ or $x^2+xy+y^2=0$

$x^2+xy+y^2 \geq 0$ for any $x, y \in \mathbb{R}\setminus\{0\}$

because if x and y have the same sign $x^2>0$, $xy>0$, $y^2>0$ so $x^2+xy+y^2>0$
and if x and y have opposite sign then $x^2+y^2 \geq 2|xy|$
so $x^2-2|xy|+y^2 \geq 0$ so $x^2-|xy|+y^2>0$.

If x or $y=0$
then we have
$0^3=y^3$ so $y=0$.
or $x^3=0$ so $x=0$
either way $x=y$.
Since $x^2+xy+y^2>0$
whenever x and y are not 0
then
$x=y$
so
f is one-to-one

   (b) [5 points] Prove or disprove: f is onto.

f is not onto because there is no $x \in \mathbb{Z}$ s.t. $f(x)=2$.

Indeed $f(0)=0$, $f(1)=1$, $f(2)=8$

and f is increasing, so

f skips the value 2.

$\boxed{f \text{ is not onto}}$

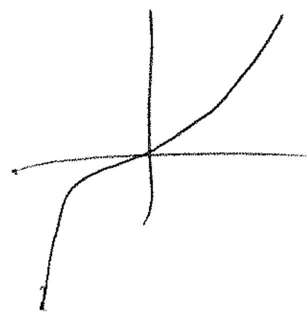
(c) [5 points] Prove or disprove: g is one-to-one.

Same proof as the one for (a).

$\boxed{g \text{ is one-to-one.}}$

(d) [5 points] Prove or disprove: g is onto.

$g(x) = x^3$ has graph



so $\boxed{g \text{ is onto}}$

(if $y \in \mathbb{R}$ $\exists$ x s.t $g(x) = y$, indeed take $x = \sqrt[3]{y}$, which exists for any y.)

5. Let $A = \{1, 2, 3, \ldots, 10\}$ and $B = \{1, 2, 3\}$.

(a) [5 points] How many functions $f : A \to B$ are there?

$\boxed{3^{10}}$

(b) [5 points] How many functions $f : A \to B$ satisfy that $f(1) = 1$, $f(2) = 2$ and $f(3) = 3$?

$\boxed{3^{7}}$

(c) [5 points] How many functions $f : A \to B$ are one-to-one?

$$\boxed{0}$$

(d) [5 points] How many functions $f : A \to B$ are onto? (BONUS)

$$3^{10} - (3)(2^{10}) + 3 \cdot 1^{10} = \boxed{55980}$$

($3^{10} = 59049$, so most function from A to B are onto with these choices of A and B)

6. Pigeonhole

(a) [5 points] Prove that if you select 1007 numbers from the set $\{1, 2, 3, \ldots, 2013\}$, at least two of the numbers will be consecutive.

$\{1, 2\}$
$\{3, 4\}$
$\vdots$
$\{2011, 2012\}$
$\{2013\}$

FALSE.

$1, 3, 5, 7, \ldots, 2013$ are 1007 numbers that are not consecutive.

(b) [5 points] 10000 fans went to see a soccer game. Prove that there are at least 28 people with the same birthday.

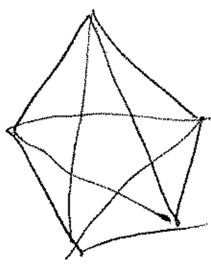
There are 366 possible birthdays.

$$
\begin{array}{r}
27 \\
366 \overline{)10000} \\
2680 \\
118
\end{array}
$$

Since $10000 > 366(27)$ then ~~by~~ the Pigeonhole Principle, there must be one birthday that at least 28 people have.

7. Diagonals of polygons

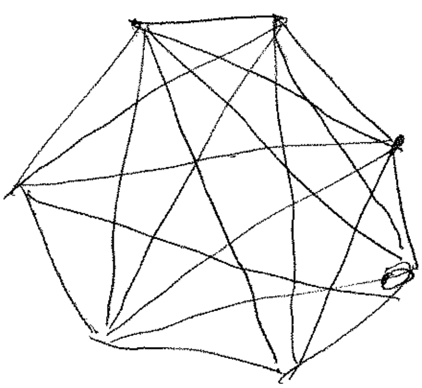
(a) [5 points] How many diagonals does a regular pentagon have? (Note: A line segment of a polygon is considered a diagonal if it is the line segment between two vertices of the polygon but not a side of the polygon. For example, a triangle has no diagonals because the line segments connecting two vertices of the triangle are the sides of the triangle, while the square has two diagonals because the line segments connecting two opposite corners of a square are not sides of the square.)



5

$\binom{5}{2} - 5 = 10 - 5 = 5$

(b) [5 points] How many diagonals does a regular heptagon have? (Note: An heptagon has 7 sides)



$\binom{7}{2} - 7 = \frac{7 \cdot 6}{2} - 7$

$= 14$

(c) [5 points] How many diagonals does a regular n-gon have? (Explain why it has the number you claim).

$$\boxed{\underbrace{\binom{n}{2}}_{\text{\# of lines connecting two vertices}} - \underbrace{n}_{\text{\# of sides.}}}$$