Geometry
Homework 1

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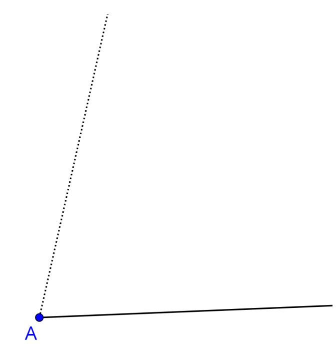
In the problems involving straightedge and compass constructions, you may take for granted the construction of perpendicular line bisectors, angle bisectors, equilateral triangles and squares. In other words, you can describe a step as "we draw the angle bisector at ___" as opposed to also describing how you find the angle line bisector. You may also take for granted that given a point P and a line ℓ, you can construct a perpendicular line to ℓ through P. Thales theorem without proof.

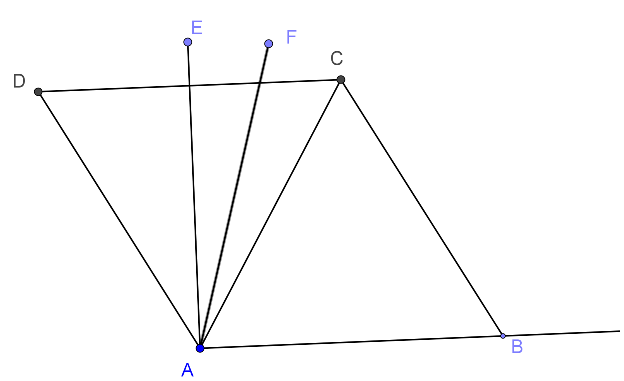
1. Given a point A and a line ℓ through A. Describe how you would create, using only straightedge and compass, a line k that goes through A that satisfies that the small angle between k and ℓ is $75°$. In the figure below, the dotted line is what k should be and ℓ is the solid line.



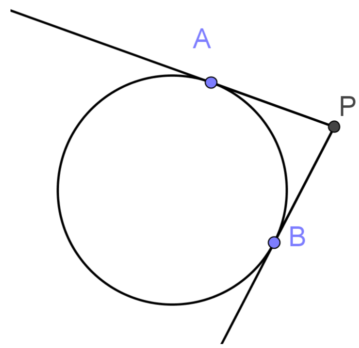
Solution 1. Let B be a point on line ℓ to the right of A. Now find C above the line such that $\triangle ABC$ is equilateral. Now find D to the left of AC such that $\triangle ADC$ is equilateral. Now bisect the angle $\measuredangle CAD$ and pick a point E in the angle bisector. Now bisect the angle $\measuredangle CAE$ and pick a point F in the angle bisector. Then

$$\measuredangle CAF = \frac{1}{2}\measuredangle CAE = \frac{1}{4}\measuredangle CAD = 15°.$$

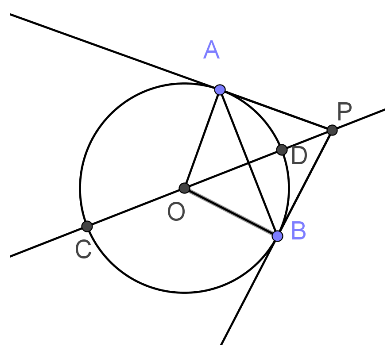
Therefore $\measuredangle BAF = \measuredangle BAC + \measuredangle ACF = 60° + 15° = 75°$. Then the line k is the line AF.



2. Given two points A and B on a circle Γ, describe how you can find, using only straightedge and compass, a point P such that the rays PA and PB are tangent to Γ.



Solution 2. The first part is to find the center of the circle. Let O be the name for the center of the circle. We know that $OA = OB$, so O has to be in the perpendicular line bisector of AB. We draw the perpendicular line bisector to AB and we let C and D be the intersections of the line with the circle. Then CD is a diameter, so O is the midpoint of CD (which we can find with straight edge and compass). Therefore, we've found the center of the circle. Now simply draw the lines perpendicular to OA and OB through A and B, respectively. These lines are the tangent lines. Their intersection is our desired P.



3. Given a regular n-gon, describe how you can find using only straightedge and compass, a regular $2n$-gon.

Solution 3. Consider a regular polygon with vertices $A_1, A_2, \ldots A_n$. We will first prove that every polygon can be inscribed in a circle. First, let α be the angle at every vertex. Then $\alpha n = 180(n-2)$ degrees (proving this is part of Homework 2). Let O be the intersection of the angle bisector of $\measuredangle A_n A_1 A_2$ with the angle bisector of $\measuredangle A_1 A_2 A_3$. Then $OA_1 = OA_2$ by construction. Furthermore, $\measuredangle A_1 O A_2 = 180 - \measuredangle OA_1A_2 - \measuredangle OA_2A_1 = 180 - 2\measuredangle OA_1A_2 = 180 - \alpha$. Since $n\alpha = (n-2)(180)$, then

$$n\measuredangle OA_2A_1 = 180n - n\alpha = 180(n-(n-2)) = 180(2) = 360. \tag{1}$$

So the angle OA_1A_2 fits exactly n times around the point O. We can draw $A_3', A_4', \ldots, A_n'$ satisfying that $OA_1 = OA_2 = OA_3' = OA_4' = \ldots = OA_n'$ and that $\measuredangle A_1OA_2 = \measuredangle A_2OA_3' = \measuredangle A_3'OA_4' = \ldots = A_{n-1}'OA_n' = A_n'OA_1$ (it lands back at A_1 because if we do it n times we end up at the beginning by (1)). By SAS we get that $A_1A_2 = A_2A_3' = A_3'A_4' = \ldots = A_{n-1}'A_n' = A_n'A_1$. Therefore the n-gon $A_1A_2A_3'A_4' \ldots A_n'$ is a regular n-gon that can be fit in a circle. We can now see that $A_3' = A_3, A_4' = A_4, \ldots, A_n' = A_n$.

The proof above not only shows that every regular polygon can be inscribed in a circle, but that the center of that circle is the intersection of the angle bisectors of two consecutive vertices. We can also conclude that all angle bisectors intersect in the center of the circle. For our construction, what we will do is find the angle bisectors of A_1 and A_2. The intersection is O. Now draw the circle Γ with center O and radius A_1. This circle will catch all vertices of the regular n-gon (as proven above). For each $i = 1, 2, \ldots, n$, find the perpendicular line bisector of A_iA_{i+1} (where we wrap around the circle by saying $i + 1 = 1$ when $i = n$). The perpendicular line bisector intersects the circle twice. Consider the intersection on the side of A_iA_{i+1} and call it A_i'. Then $A_iA_i' = A_i'A_{i+1}$ by construction. This is true all around the circle, so the $2n$-gon $A_1A_1'A_2A_2' \ldots A_nA_n'$ is regular.

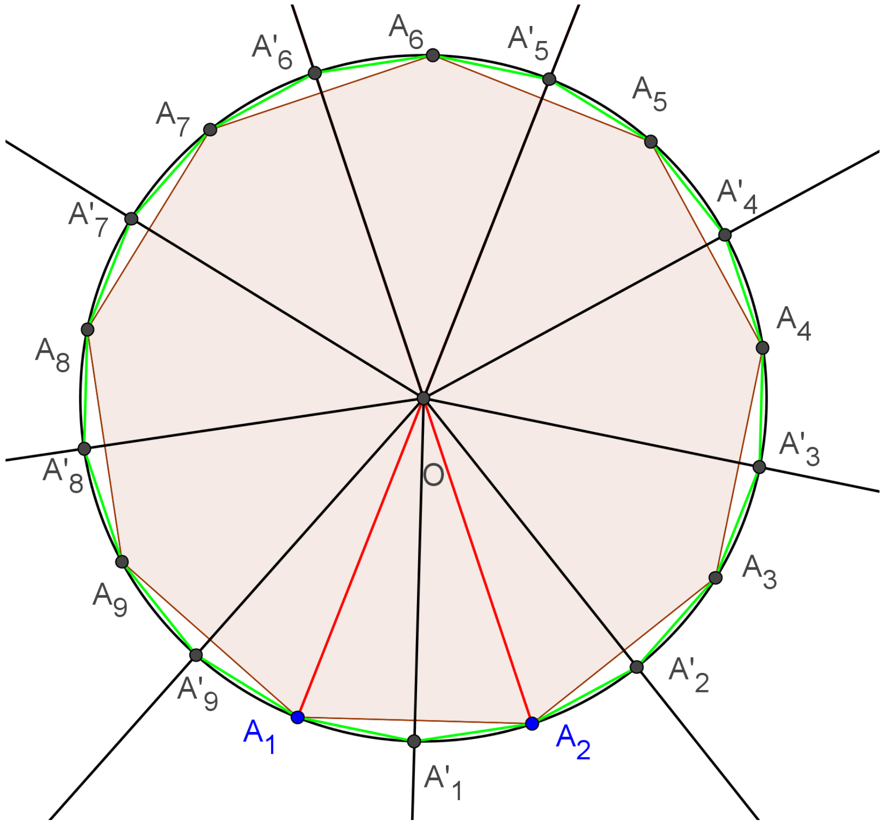
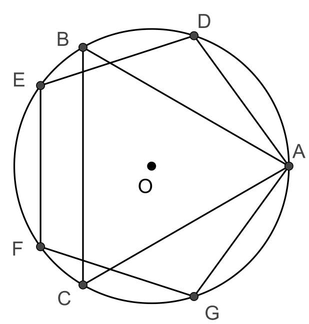


Figure 1: The construction of an 18 sided regular polygon from a nine sided regular polygon. The read lines are the angle bisectors of angles $\measuredangle A_1$ and $\measuredangle A_2$, while the black rays (that go through O) are the perpendicular line bisectors of the edges of the polygon. The 18-sided polygon would be the polygon $A_1A_1'A_2A_2'\ldots A_9A_9'$ (the edges are green).

4. Given a regular n-gon and a regular m-gon satisfying that n and m are relatively prime[1], show that you can create a regular nm-gon using only straightedge and compass.

   Solution 4. In class we described how we can shrink or expand the polygons and move them so that they both fit in a unit circle and they both share a vertex A (all done with straightedge and compass). I will now describe how to do this for the example of the triangle and a pentagon and then I will explain how it generalizes. First place the polygons in a unit circle. Let ABC be an equilateral triangle and $ADEFG$ be a regular pentagon as in the figure below:



   A regular n-gon with adjacent vertices A, B satisfies that $\measuredangle AOB = \frac{360}{n}$, where O is the center of the circle. Now in the figure above consider the angle $\measuredangle DOB$. We have

$$\measuredangle DOB = \measuredangle BOA - \measuredangle DOA = \frac{360}{3} - \frac{360}{5} = \frac{2}{15}(360).$$

[1]this means that there is no integer $d > 1$ such that $d|n$ and $d|m$

Now by using the length BD we can place the compass at B with radius BD and find point H on the circle such that $BH = DH$. We can then place the compass on H with radius HB and find that the compass lands on F (which we already had). We continue around the circle in this fashion. After 15 steps we'll end up back at B. So we'll have 15 points $A, Q, K, D, L, B, E, H, N, F, C, I, G, J, P$, all equidistant across the unit circle. So we have a regular 15-gon. The picture below illustrates the process:

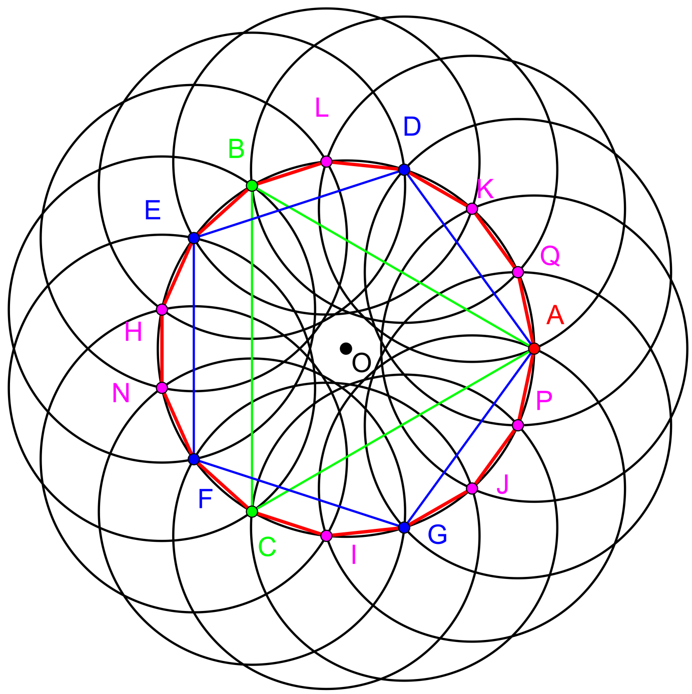


Figure 2: ABC is an equilateral triangle on the unit circle, $ADEFG$ is a regular pentagon on the unit circle. By drawing circles of radius BD around the circle we find the points that build the regular 15-gon on the unit circle.

Now, why does this work? The key is that the angle DOB is 2/15 of 360, so if we move with radius BD we create angles of that size. After making 15 of them, they break the circle in the middle with 15 angles of length 360/15. This works because 2 and 15 are relatively prime, so they don't have any factors in common. Note that if you try this with 3/15, you don't eventually break down into fifteen angles, after 5 circles you are back at the beginning ($3 \times 5 = 15$).

The generalization goes as follows. Suppose $A_1A_2 \dots A_n$ is a regular n-gon on the unit circle and that $B_1B_2 \dots B_m$ is a regular m-gon on the unit circle with $A_1 = B_1$. One of n and m is bigger, so suppose that $m > n$. Now consider B_2A_2 and in particular the central angle opening this chord.

$$\measuredangle A_2OB_2 = \measuredangle A_2OA_1 - \measuredangle B_2OB_1 = \frac{360}{n} - \frac{360}{m} = \frac{m-n}{mn}(360)ll.$$

We can then draw circles with radius B_2A_2 by moving from point to point and do it mn times. We know that m and n are relatively prime. Therefore $\gcd(m, n) = 1$. Now $\gcd(m-n, n) = \gcd((m-n)+n, n) = \gcd(m, n) = 1$, and $\gcd(m-n, m) = \gcd(-n, m) = \gcd(n, m) = 1$. Therefore $\gcd(m-n, mn) = 1$. That means that the circles don't cycle before one draws mn circles, so we will have mn equidistant points around the circle and we can build the regular mn-gon now.

5. Exercise 1.3.5 from the book.

Solution 5. Here's a proof that trisecting a segment does not trisect an angle. Let $OA'B'$ be a triangle with $\measuredangle A'OB' = 90°$. Now let prolong $A'B'$ in the direction of A' to find A such that $AA' = A'B'$. Similarly prolong $A'B'$ in the direction of B' to find B such that $BB' = A'B'$. Then A' and B' trisect AB, but OA' and OB' do not trisect $\measuredangle AOB$ because if it did trisect it, then $\measuredangle A'OB' = \frac{1}{3}\measuredangle AOB < \frac{180°}{3} = 60°$. But $\measuredangle A'OB' = 90°$, so we have a contradiction. Therefore trisecting a segment is not enough to trisect an angle.

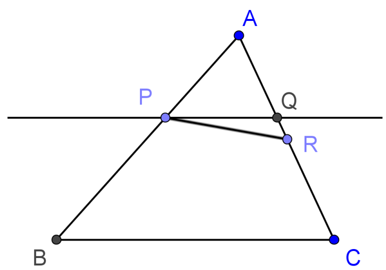
6. Exercises 1.3.6, 1.4.1 and 1.4.2.

   Solution 6. For 1.3.6: Suppose $\frac{AP}{PB} = \frac{AQ}{QC}$. Then

$$\frac{AB}{AP} = \frac{AP + BP}{AP} = 1 + \frac{BP}{AP} = 1 + \frac{CQ}{AQ} = \frac{AQ + QC}{AQ} = \frac{AC}{AQ}.$$

   Therefore $\frac{AP}{AB} = \frac{AQ}{AC}$.

   For 1.4.1: Draw the parallel line to AB through P. Let Q be the intersection of this parallel line with AC. Since $PR \not\parallel AB$, then Q and R are different points. Now we know that $AP/AB = AQ/AC$. But we also have that $AP/AB = AR/AC$. Then $AR = AQ$. But since R and Q are on the line segment AC and $AR = AQ$, we have that $R = Q$. This contradicts the assumption that $AR \not\parallel AB$.



   For 1.4.2: Suppose P is any point on AB and Q is any point on AC. If $PQ \| AB$, then by Thales (which we proved later in chapter 2) $\frac{AP}{PB} = \frac{AQ}{QC}$. But then by 1.3.6 this means $\frac{AP}{AB} = \frac{AQ}{AC}$. Therefore $PQ \| AB \Rightarrow \frac{AP}{AB} = \frac{AQ}{AC}$.

   Now suppose P is any point on AB and Q is any point on AC and $\frac{AP}{AB} = \frac{AQ}{AC}$. Then by 1.4.1, that means $PQ \| AB$. Therefore the if and only if statement has been proved.

7. Exercises 1.4.3 and 1.4.4.

   Solution 7. For 1.4.3: Since $AB \| ED$, then

$$\frac{OA}{OE} = \frac{OB}{OD}.$$

   Since $FE \| BC$, then

$$\frac{OE}{OC} = \frac{OF}{OB}.$$

   Now multiply these two equation to get

$$\frac{OA}{OE} \cdot \frac{OE}{OC} = \frac{OB}{OD} \cdot \frac{OF}{OB}$$
$$\frac{OA}{OC} = \frac{OF}{OD}.$$

   Therefore

$$\frac{OA}{OF} = \frac{OC}{OD}.$$

   Now, by Thales we get that $AF \| CD$.

   For 1.4.4: Since $AB \| A'B'$, then

$$\frac{OA}{OA'} = \frac{OB}{OB'}.$$

   Since $BC \| B'C'$, then

$$\frac{OB}{OB'} = \frac{OC}{OC'}.$$

   Therefore $\frac{OA}{OA'} = \frac{OC}{OC'}$, and hence

$$\frac{OA}{OC} = \frac{OA'}{OC'}.$$

   Now, by Thales we get that $AC \| A'C'$.

8. Exercises 1.5.1, 1.5.2, 1.5.3 and 1.5.4.

Solution 8. 1.5.1:
$$\frac{\sqrt{2}+1}{1} = \frac{(\sqrt{2}+1)(\sqrt{2}-1)}{\sqrt{2}-1} = \frac{2^2-1^2}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1}.$$

1.5.2: Suppose $a/b = \sqrt{2}+1$. Then $a = (\sqrt{2}+1)b$. Therefore
$$a - 2b = (\sqrt{2}+1)b - 2b = (\sqrt{2}-1)b.$$

Then
$$\frac{b}{a-2b} = \frac{b}{(\sqrt{2}-1)b} = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 = \frac{a}{b}.$$

1.5.3: Suppose $\sqrt{2}+1 = m/n$ with m and n positive integers with m as small as possible. By 1.5.2 we have that if $\sqrt{2}+1 = m/n$, then $\sqrt{2}+1 = n/(m-2n)$. Since n is positive and $\sqrt{2}+1$ is positive, then $m-2n$ is positive. So we have $\sqrt{2}+1 = n/(m-2n)$ where n and $m-2n$ are positive integers. Furthermore, since $\sqrt{2}+1 > 1$, then $m > n$. Therefore we have a representation of $\sqrt{2}+1$ as a ratio of two positive integers with a numerator smaller than m. That contradicts our choice of m. Therefore $\sqrt{2}+1$ cannot be the ratio of two integers.

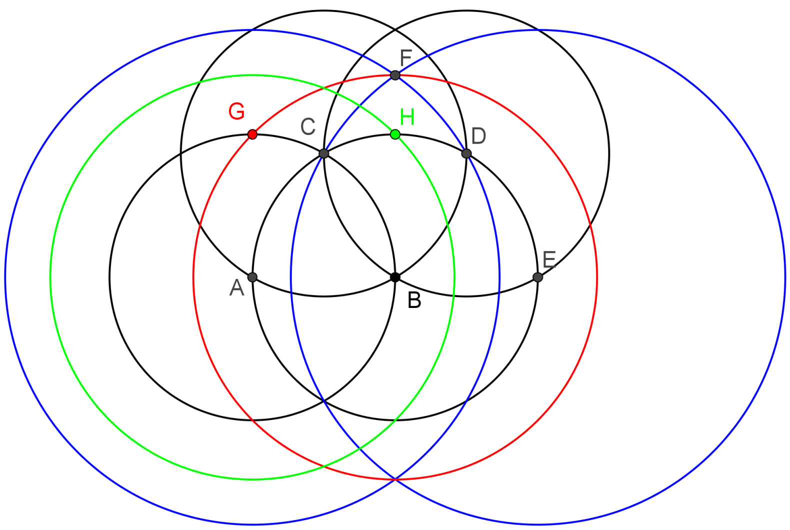
1.5.4: Suppose that $\sqrt{2}$ is rational, i.e., that there exist integers p and q such that $\sqrt{2} = p/q$. But then
$$\sqrt{2}+1 = \frac{p}{q} + 1 = \frac{p+q}{q}.$$

Therefore $\sqrt{2}+1$ is the ratio of two integers and hence rational. But we know $\sqrt{2}+1$ is irrational, so we've reached a contradiction. Therefore $\sqrt{2}$ is irrational.

BONUS In class I mentioned that given two points A and B, one can find using only compass (without the straightedge) a point C such that $\triangle ABC$ is equilateral. One can also find points to make an hexagon using only compass. Prove or disprove that you can find, using only a compass, points C and D such that $ABCD$ is a square.

Solution 9. Begin with A and B. Now find C that makes ABC equilateral by intersecting the circle with radius AB centered at A with the circle with radius AB centered at B. Then find D to make BCD equilateral and E to make BDE equilateral as in the figure below. Now we have that A, B, E are collinear and that $AB = BE$. We also have that $CE = \sqrt{3}AB = AD$ because they are twice the height of an equilateral triangle of length AB. Now draw the circle of radius CE centered at E and the circle of radius AD centered at A (in the figure below they are colored blue). Let F be their intersection. Since $AF = EF$, then the line from F to the midpoint of AE is perpendicular to AE. But B is the midpoint of AE. Therefore $BF \perp AE$. Now $AF = \sqrt{3}AB$, so by Pythagoras we have
$$BF^2 = AF^2 - AB^2 = 3AB^2 - AB^2 = 2AB^2.$$

Therefore $BF = \sqrt{2}AB$. Now draw the circle of radius BF with center at B (colored red) and intersect it with the circle of radius AB centered at A. Call this intersection G. Then we have that $BG = \sqrt{2}AB$ and $AG = AB$, so $BG^2 = 2AB^2 = AB^2 + AB^2 = AB^2 + AG^2$. Therefore, by the converse of the Pythagorean theorem $\angle BAG = 90°$. Since $AG = AB$ and $AG \perp AB$, we have one of the two points we need to find to get the square. We can build the other point, which we'll call H, similarly and get the square (I drew the final circle that would be drawn for the other side in green). Then $ABHG$ is a square.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