Geometry
Homework 2 Solutions

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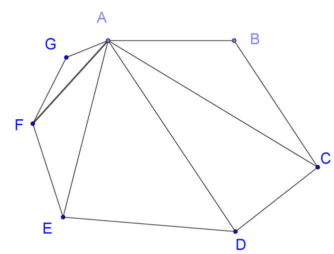
September 10, 2016

1. Exercises 2.1.2, 2.1.3, 2.1.4 and 2.1.5.

Solution 1. For exercise 2.1.2, consider the following heptagon $ABCDEFG$:



By drawing all of the diagonals that spring out from vertex A we triangulated the heptagon. There are 5 triangles. It's easy to see why this generalizes. Suppose you have the n-gon $A_1A_2\ldots A_n$. Now draw all of the diagonals that spring out from vertex A_1. Since A_1A_2 and A_1A_n are edges of the n-gon, the diagonals have the form A_1A_i for $i = 3, 4, 5, \ldots, n-1$. Therefore we have $n-3$ diagonals. Note that as we draw the diagonals in order we form one triangle each time except for the last diagonal which splits $A_1A_{n-2}A_{n-1}A_n$ in two triangles. Therefore we have $n-2$ triangles forming the n-gon.

For exercise 2.1.3 note that the process from the previous paragraph triangulated an n-gon into $n-2$ triangles whose angles are parts of the original angles of the n-gon. It is easy to see that the sum of the angles of the n-gon is equal to the sum of the angles of the $n-2$ triangles. But since the angles in each triangle add up to $180°$, then the sum of the angles of the polygon add up to $180°(n-2)$ or $(n-2)\pi$ radians.

For exercise 2.1.4 note that a regular n-gon has all of its angles equal to each other. Therefore the angle at each vertex is $\frac{n-2}{n}(180°)$ or $\frac{n-2}{n}\pi$ radians.

For exercise 2.1.5 note that if you tile the plane with regular n-gons, then you have to be able to glue together many copies of the same n-gon. Since the angles that are glued together add up to $360°$, then you can only use n-gons whose angles are divisors of $360°$. But then $360 = 180\left(\frac{n-2}{n}\right)k$ for some integer k. So $2n = (n-2)k$. This implies $n(k-2) = 2k$. But then

$$n = \frac{2k}{k-2} = 2\frac{k}{k-2} = 2\left(1 + \frac{2}{k-2}\right) = 2 + \frac{4}{k-2}.$$

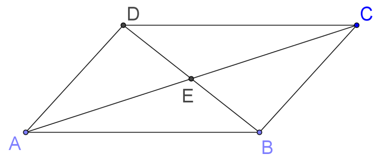
So $k-2$ has to divide 4. There are only three possibilities $k-2=1$, $k-2=2$, or $k-2=4$. These values of k correspond to $n = 6, 4,$ and 3 respectively. Therefore you can only tile the plane with triangles, squares or hexagons (if you restrict yourself to regular polygons).

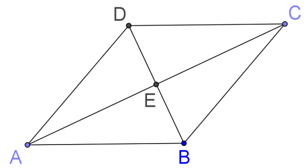
2. Exercises 2.2.1, 2.2.2 and 2.2.3.

Solution 2. For Exercise 2.2.1 consider the figure with the following labels:

Since $AB \| CD$ we have that $\measuredangle CDB = \measuredangle DBA$ and $\measuredangle DCA = \measuredangle CAB$. Since $ABCD$ is a parallellogram, then $AB = CD$, so by ASA we have that $\triangle CDE \cong \triangle ABE$. Therefore $DE = BE$ and $AE = CE$, so the diagonals of the parallellogram bisect each other.

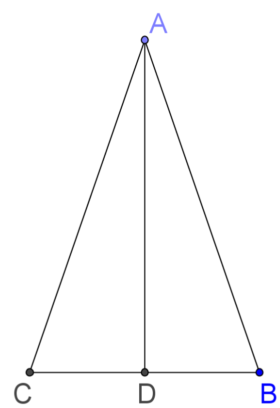
For exercise 2.2.2 use the following diagram:





Note that a rhombus is also a parallellogram, so its diagonals bisect each other. Since all the sides of a rhombus are equal, then in particular $AB = BC$. Since the diagonals bisect each other, then $AE = CE$. Now $\triangle ABE$ and $\triangle CBE$ have two equal sides and share the side BE, therefore $\triangle ABE \cong \triangle CBE$. Therefore $\measuredangle AEB = \measuredangle CEB$. Since $\measuredangle AEB + \measuredangle CEB = 180°$, then $\measuredangle AEB = \measuredangle CEB = 90°$. Therefore $AC \perp BD$.

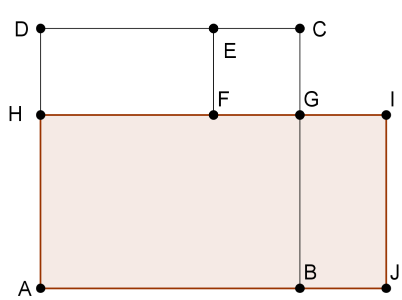
For exercise 2.2.3 we'll use the following figure, where AD is the angle bisector of $\measuredangle BAC$:



By construction we have $\measuredangle CAD = \measuredangle BAD$. We also have that $AB = AC$. Since $\triangle ABD$ and $\triangle ACD$ share the side AD, then by SAS we have $\triangle ABD \cong \triangle ACD$. Therefore $\measuredangle ABD = \measuredangle ACD$, which implies that $\measuredangle ABC = \measuredangle ACB$, which is what we wanted to prove.

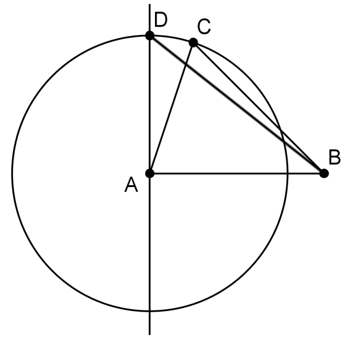
3. Exercise 2.3.3.

Solution 3. In the following figure let $AB = BC = CD = AD = a$ and $CG = GF = FE = CE = b$. Then the area represented by $ABGFEDA$ is $a^2 - b^2$ because it is the area of $ABCD$ minus the area of $CEFG$. Now extend AB and HG to points J and I, respectively, in such a way that $BJ = GI = b$, which is also EF. Then $AH = IJ = BG = a - b$. But HF is also $a - b$. Therefore the rectangle $BJIG$ is congruent to the rectangle $HFED$. Therefore the area of the rectangle $AJIH$ is $a^2 - b^2$. But the base of the rectantle is $AJ = a + b$ and the height of the rectangle is $IJ = a - b$. Therefore $a^2 - b^2 = (a + b)(a - b)$.



4. Exercises 2.5.2 and 2.5.3.

Solution 4. For 2.5.2: Suppose that you have a triangle ABC with sides a, b, c such that $b^2 + c^2 = a^2$ and that the triangle is not a right triangle. Consider the perpendicular to AB through A. Now intersect this perpendicular line with the circle centered at A with radius AC. Let D be the intersection above AB. Then $AD = AC = b$ and $AD \perp AB$. Since DAB is a right triangle and $AB = c$ and $AD = b$, then by the Pythagorean Theorem $AD^2 + AB^2 = BD^2$, so $BD^2 = b^2 + c^2 = a^2$. Therefore $BD = a$. But then $\triangle DAB \cong \triangle CAB$ by SSS. Therefore $\measuredangle DAB = \measuredangle CAB = 90°$. This contradicts that $\triangle CAB$ is not a right triangle.



For 2.5.3: Suppose $a, b, c > 0$ and $a^2 + b^2 = c^2$. We want to show that there exist a triangle with lengths a, b, c. We need to show that $a + b > c, a + c > b$, and $b + c > a$. Since $a^2 + b^2 = c^2$ and $a, b > 0$, then $c^2 > a^2$, and $c^2 > b^2$. So $c > a$ and $c > b$. Then $b + c > a + b > a$, and $a + c > a + b > b$. Therefore we have two of the three inequalities we need just from the fact that c is the biggest side of a, b, c. The tricky inequality is showing that $a + b > c$. For the sake of contradiction suppose that $a + b \leq c$. Then $(a + b)^2 \leq c^2$, so

$$\begin{aligned} a^2 + 2ab + b^2 &\leq c^2 \\ a^2 + 2ab + b^2 &\leq a^2 + b^2 \\ 2ab &\leq 0. \end{aligned}$$

But this is impossible. Therefore $a + b > c$. We have proved that there is a triangle with such lengths.

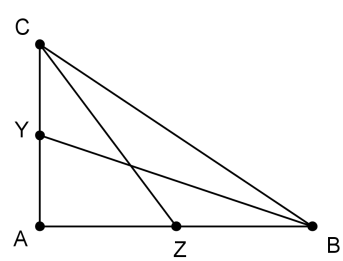
5. Exercises 2.5.4 and 2.5.5.

Solution 5. For 2.5.4: Draw a line of length 1 with vertices A, B. Now draw a perpendicular line to AB through A. Find a point C in this perpendicular line such that $AC = \sqrt{2}$. Then by the Pythagorean theorem, $BC = \sqrt{3}$.

For 2.5.5: Suppose that you have built 1 and $\sqrt{n}$, let's built $\sqrt{n+1}$. Do the same process as above but make $AC = \sqrt{n}$. Then by Pythagoras, $BC = \sqrt{n+1}$. Therefore, by induction we can make all lengths $\sqrt{n}$ for any natural number n.

6. Let ABC be a right triangle with $\angle A = 90°$. Let Y and Z be the midpoints of segments AC and AB, respectively. Let $BY = \sqrt{73}$ and $CZ = 2\sqrt{13}$. Find the length of BC.

Solution 6. Let $AB = c$ and $AC = b$. Then $AZ = c/2$ and $AY = b/2$. By Pythagoras we have:



$$52 = (2\sqrt{13})^2 = CZ^2 = AC^2 + AZ^2 = b^2 + \frac{c^2}{4},$$

$$73 = (\sqrt{73})^2 = BY^2 = AY^2 + AB^2 = \frac{b^2}{4} + c^2.$$

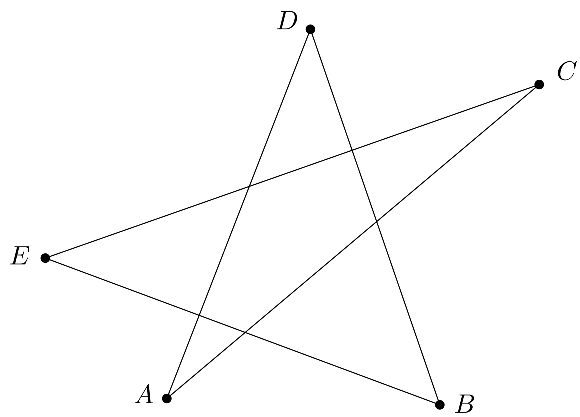
Adding these two equations we get

$$125 = \frac{5}{4}\left(b^2 + c^2\right) = \frac{5}{4}BC^2.$$

Therefore $BC^2 = 100$, so $BC = 10$.

7. Let $ABCDE$ be a (not necessarily regular) five point star. Find the sum (with proof)

$$\angle A + \angle B + \angle C + \angle D + \angle E.$$



Solution 7. Starting with the intersection of AD with BE, label the points in the inner pentagon clockwise as F, G, H, I, J. Consider the triangle BFD. Then

$$\measuredangle BFD = 180^\circ - \angle B - \angle D.$$

Similarly

$$\begin{aligned} \measuredangle AGC &= 180^\circ - \angle A - \angle C, \\ \measuredangle EHB &= 180^\circ - \angle E - \angle B, \\ \measuredangle DIA &= 180^\circ - \angle D - \angle A, \\ \measuredangle CJE &= 180^\circ - \angle C - \angle E. \end{aligned}$$

Adding these 5 equations we get that the sum of the interior angles of the pentagon is

$$5(180) - 2(\angle A + \angle B + \angle C + \angle D + \angle E).$$

But we know that the angles of a pentagon add up to $(5-2)(180) = 3(180) = 540$. Therefore

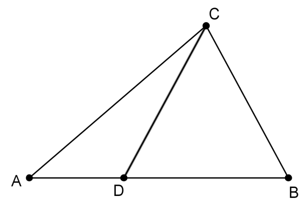
$$540 = 900 - 2(\angle A + \angle B + \angle C + \angle D + \angle E).$$

Therefore

$$\angle A + \angle B + \angle C + \angle D + \angle E = \frac{900 - 540}{2} = \frac{360}{2} = 180.$$

8. Prove or disprove: For triangles ABC and $A'B'C'$ we know that $AB = A'B'$, $AC = A'C'$ and $\angle BCA = \angle B'C'A'$. Then they must be congruent.

Solution 8. They are not necessarily congruent. Indeed consider the figure below. $\triangle ACD$ has sides AC and $AD = BC$, and it also has the angle $\measuredangle CAD = \measuredangle CAB$. Now $\triangle ACB$ has sides AC and AB and the angle $\measuredangle CAB$. So $\triangle ACD$ and $\triangle ACB$ have two equal sides and an equal angle, yet they are not congruent to each other.



BONUS What is the least possible area of a triangle $\triangle ABC$ with altitudes satisfying $h_a \geq 3$, $h_b \geq 4$, $h_c \geq 5$? Note: h_a is the height of the triangle when BC is the base, h_b is the height when AC is the base, and h_c is the height when AB is the base.

Solution 9. First let's consider right triangles. If the base of the triangles are $h_a = 3$ and $h_b = 4$, then the hypotenuse is 5 and then by areas we have $3 \times 4 = 5 \times h_c$, so $h_c = 12/5 < 5$. So this one doesn't work.

If the base of the triangles are $h_a = 3$ and $h_c = 5$, then the hypotenuse is $\sqrt{34}$. Using areas we have $3 \times 5 = \sqrt{34} h_b$. So $h_c = 15/\sqrt{34} < 4$ because $16 \times 34 = 544 > 225 = 15^2$.

The last option is a right triangle with bases $h_b = 4$ and $h_c = 5$. Then $4 \times 5 = h_a \times \sqrt{41}$. Therefore $h_a = \frac{20}{\sqrt{41}} > 3$ because $9 \times 41 = 369 < 400 = 20^2$. Therefore one possible triangle is a right triangle with bases 4 and 5. This triangle has area 10.

Now we will prove that 10 is the smallest possible area. Consider the height $h_b \geq 4$. Since AB is a hypotenuse for the triangle with height AB, then $c = AB \geq 4$. But then the area of ABC is

$$(ABC) = \frac{c \times h_c}{2} \geq \frac{c \times 5}{2} \geq \frac{4 \times 5}{2} = 10.$$

Therefore the area is always at least 10 and we created a triangle that works with area 10. Hence the minimal area is 10.