Geometry
Homework 3 Solutions

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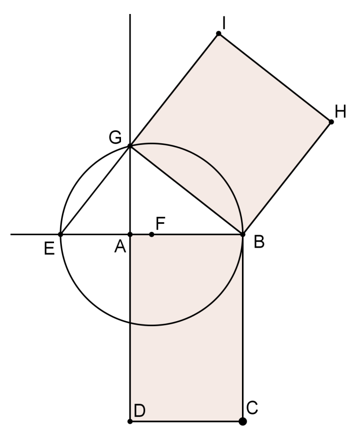
1. Exercises 2.7.2, 2.7.3, 2.7.4.

   Solution 1. For 2.7.2: Start with a rectangle $ABCD$. Now prolong the line BA in the direction of A and choose a point E. Now let F be the midpoint of E and B. Draw the circle Γ centered at F with radius BF. Also prolong DA upwards. Let G be the intersection of DA and Γ. Now draw the square of side length BG. Then the square $BGIH$ has the same area as the rectangle $ABCD$ (proven in the proof of the Pythagorean theorem).



   For 2.7.3: Suppose you have a square of side length a and a square of side length b. Now build a right triangle with base b and height a. Then the hypotenuse satisfies that $a^2 + b^2 = c^2$. Therefore, the square formed with side length equal to the hypotenuse has the same area as the sum of the area of the square with side length a plus the square with side length b.

   For 2.7.4: Suppose you have an n-gon. Then you can cut the n-gon into $n-2$ triangles. Let's call them $T_1, T_2, \ldots, T_{n-2}$. Let b_i and h_i be the base and height of triangle T_i, respectively. Then the area of T_i is the same as the area of a rectangle with base b_i and height $h_i/2$. Therefore we can represent it as the area of a rectangle. But as shown in 2.7.2, we can then translate this into the area of a square. Therefore, the $n-2$ triangles can be transformed (with straightedge and compass) into $n-2$ squares $S_1, S_2, \ldots, S_{n-2}$ where the area of S_i is the same as the area of T_i. To complete the proof we will need to use 2.7.3 multiple times. Since the sum of two squares can be represented by one square, then we can pair up two squares and have one. At first we replace $S_1 + S_2$ with S_1'. Now replace $S_1' + S_3$ with S_2'. Continue until you are left with one square at the end (if you keep labelling the same way, the last one will be called S_{n-3}'). This last square has the same area as the sum of the areas of the original triangles, which is the same as the area of the original n-gon.

2. Exercises 2.8.1, 2.8.2 and 2.8.3.

   Solution 2. For 2.8.1: Consider the following figure:

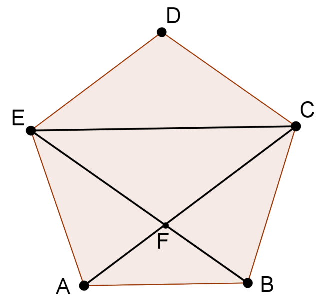


Figure 1: Pentagon with side length 1 and diagonals of length x

Since $DE = AB$, $CD = BC$ and $\measuredangle ABC = \measuredangle CDE$, then $\triangle CDE \cong \triangle CBA$. Then $\measuredangle CED = \measuredangle BAC$. Therefore $EC \| AB$. Then by alternate interior angles $\measuredangle CEB = \measuredangle EBA$ and $\measuredangle ECA = \measuredangle CAB$. Therefore $\triangle CEF \sim \triangle ABF$. This implies that

$$\frac{CE}{AB} = \frac{EF}{BF}. \tag{1}$$

Now since $BC = AB$, then $\measuredangle ACB = \measuredangle BAC$. But $\measuredangle BAC = \measuredangle ACE$. But $\measuredangle BAC = \measuredangle CED$. Therefore $\measuredangle ACE = \measuredangle CED$. Analogously $\measuredangle BEC = \measuredangle ECD$. Using the common side EC we can conclude by ASA that $\triangle ECD \cong \triangle ECF$. Therefore $EF = ED = 1$. Since $EB = EC$ by symmetry, we have that $BF = x - 1$.

Now equation (1) translates to

$$\frac{x}{1} = \frac{1}{x-1}.$$

Therefore $x^2 - x - 1 = 0$.

For 2.8.2: By the quadratic formula we have

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Since $1 - \sqrt{5} < 0$, then x must be $\frac{1+\sqrt{5}}{2}$.

For 2.8.3: The following diagram is a construction of a pentagon of side length $AB = 1$ using only straightedge and compass:

The construction goes as follows. Extend AB. Now find C such that $AB = BC$ (use the compass with center at B and radius AB). Now draw the perpendicular line to AB through C. Now find D such that $CD = 1$. By Pythagoras we have that $AD = \sqrt{1+2^2} = \sqrt{5}$. Extend AD and pick a point E such that $DE = 1$ (use the compass centered at D with radius DC). Then $AE = 1 + \sqrt{5}$. Find the midpoint of AE and call it F. Then $AF = \frac{1+\sqrt{5}}{2}$. Draw the circle centered at A of radius AF and call it Γ_1. Intersect it with the circle centered at B of radius AB. Let the intersection be G. Then $BG = 1$ and $AG = \frac{1+\sqrt{5}}{2}$. This is the main step in the construction since we have built the points equivalent to CED in Figure 1. One could proceed similarly around the pentagon to find H and I. However, Figure 2 proceeds in a more economical fashion (with less vertices and circles). The rest of the construction goes as follows:

The diagonal of a pentagon appears twice for each vertex, so Γ_1 will intersect twice with the pentagon. We can draw a circle centered at G with radius 1 and the intersection with Γ_1 will create another point of the pentagon (which we'll call H). Now, thinking from the point of view of G, we have only drawn the diagonal GA. If we draw the circle of radius GA centered at G and intersect it with the circle centered at A of radius AB we get the final point I (this creates the diagonal GI). Now we have the five vertices that form a pentagon.

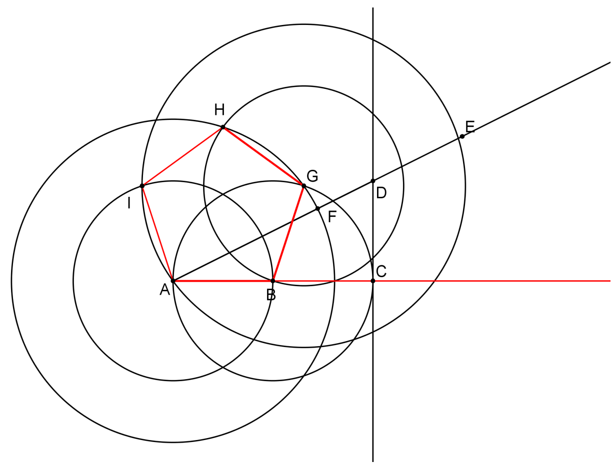
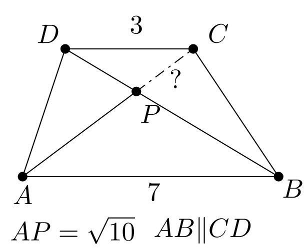
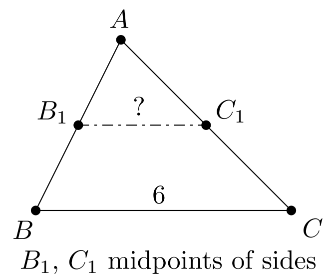
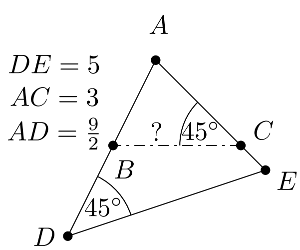


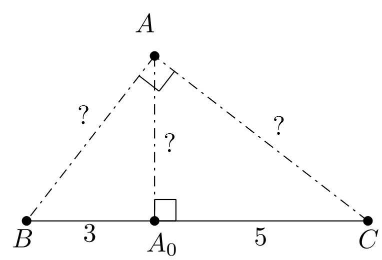
Figure 2: Straightedge and compass construction of a pentagon of side length 1

3. Find the distances denoted by question marks in the given diagrams.









Solution 3. In the top left figure: Since $AB \| CD$, then $\measuredangle DCA = \measuredangle PAB$ and $\measuredangle CDB = \measuredangle DBA$. Therefore $\triangle DCP \sim \triangle BAP$. Then

$$\frac{PC}{AP} = \frac{CD}{AB} = \frac{3}{7}.$$

Therefore

$$PC = \frac{3}{7}AP = \frac{3}{7}\sqrt{10}.$$

In the top right figure: Since B_1 and C_1 are midpoints, then $\frac{AB_1}{AB} = \frac{AC_1}{AC} = \frac{1}{2}$. Then, by Thales, $B_1C_1 \| BC$. Therefore $\triangle AB_1C_1 \sim \triangle ABC$. Therefore

$$\frac{B_1C_1}{BC} = \frac{1}{2}.$$

Then $B_1C_1 = 3$.

In the bottom left figure: $\measuredangle ACB = \measuredangle ADE$. Then $\triangle ACB \sim \triangle ADE$ (note that they share the angle $\measuredangle EAD$ and they have an equal angle). Then

$$\frac{BC}{AC} = \frac{ED}{AD} = \frac{5}{\frac{9}{2}} = \frac{10}{9}.$$

Therefore

$$BC = \frac{10}{9} AC = \frac{30}{9} = \frac{10}{3}.$$

In the bottom right figure: We have

$$\triangle ABC \sim \triangle A_0BA \sim \triangle A_0AC.$$

Therefore

$$\frac{A_0B}{AB} = \frac{AB}{BC}.$$

Then $AB^2 = A_0B \times BC = 3 \times 8 = 24$. Therefore $AB = \sqrt{24}$.

We also have

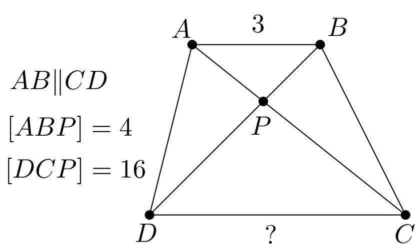
$$\frac{A_0C}{AC} = \frac{AC}{BC}.$$

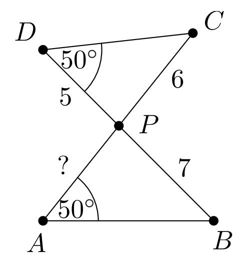
Then $AC^2 = A_0C \times BC = 5 \times 8 = 40$. Therefore $AC = \sqrt{40}$.

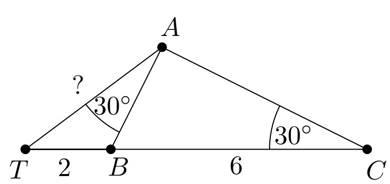
By area of the triangle we have $A_0A \times BC = AB \times AC$. Therefore

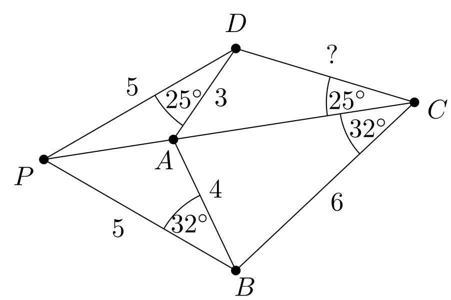
$$A_0A = \frac{\sqrt{24}\sqrt{40}}{8} = \sqrt{15}.$$

4. Find the distances denoted by question marks in the given diagrams.









Solution 4. Figure in the top left: Since $AB \| CD$, then $\triangle ABP \sim \triangle CDP$. Then

$$\left(\frac{CD}{AB}\right)^2 = \frac{(DCP)}{(ABP)} = \frac{16}{4} = 4.$$

Then $CD = \sqrt{4(AB)^2} = 6$.

Figure in the top right: Since $\measuredangle CDP = \measuredangle PAB$ and $\measuredangle CPD = \measuredangle BPA$, then $\triangle DCP \sim \triangle ABP$. Therefore

$$\frac{AP}{PD} = \frac{BP}{PC} = \frac{7}{6}.$$

Then $AP = 5 \times \left(\frac{7}{6}\right) = \frac{35}{6}$.

Figure in bottom left: $\measuredangle TAB = \measuredangle ACB$ and $\triangle ABT$ and $\triangle ACT$ share the angle $\measuredangle ATC$. Therefore, $\triangle TAB \sim \triangle TCA$. Therefore

$$\frac{AT}{BT} = \frac{TC}{AT}.$$

Then $AT^2 = TC \times BT = 8 \times 2 = 16$. Therefore $AT = 4$.

Figure in bottom right: $\triangle ADP \sim \triangle DCP$ so

$$\frac{CD}{CP} = \frac{AD}{PD} = \frac{3}{5}.$$

Then $CD = \frac{3}{5}PC$.

We also have that $\triangle PBA \sim \triangle PCB$, so

$$\frac{PC}{BC} = \frac{PB}{AB} = \frac{5}{4}.$$

Then

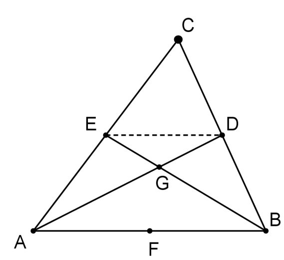
$$PC = BC \times \left(\frac{5}{4}\right) = \frac{30}{4} = \frac{15}{2}.$$

Therefore

$$CD = \frac{3}{5}PC = \left(\frac{3}{5}\right)\left(\frac{15}{2}\right) = \frac{9}{2}.$$

5. In a triangle ABC, a median is a line from a vertex to the midpoint of the opposite side. Prove that the three medians of $\triangle ABC$ intersect at a point G. Furthermore, show that if the medians are AD, BE, CF, then $AG = 2GD$, $BG = 2GE$, and $CG = 2GF$.

Solution 5. Let D, E, F be the midpoints of BC, AC, and AB, respectively. Consider the intersection of AD with BE and call it G.



Since E is the midpoint of AC and D is the midpoint of BC then $AE/AC = 1/2 = BD/BC$. Therefore by Thales, $ED \| AB$. But then $\triangle EDG \sim \triangle BAG$. Therefore

$$\frac{AG}{GD} = \frac{BG}{GE} = \frac{AB}{ED} = 2.$$

This means that $AG = 2GD$ and $BG = 2GE$.

Now suppose that the intersection of AD with CF is G'. Then by analogous methods we have $AG'/G'D = CG'/G'F = 2$. But then G and G' are points on AD satisfying that $AX/XD = 2$. There is only one point that is 2/3of the way from A to D. Therefore $G = G'$. Therefore the three medians intersect at a point G and $AG = 2GD, BG = 2GE$, and $CG = 2GF$.

6. Let $A_1 \ldots A_n$ be a regular n-gon. Find the inscribed angles corresponding to the following arcs (shorter ones):

(a) $n = 4$, A_1A_2 (b) $n = 5$, A_2A_4 (c) $n = 6$, A_1A_4
(d) $n = 12$, A_3A_7 (e) $n = 8$, A_1A_4 (f) $n = 45$, A_2A_{13}

Solution 6. Suppose you have a regular n-gon and we want to find the inscribed angle for the arc A_iA_j (with $i < j$). For A_iA_{i+1} the central angle is $\frac{360}{n}$ (since the center of the polygon is cut in n pieces). Then the central angle of the arc A_iA_j is $\frac{360(j-i)}{n}$. So the inscribed angle is $\frac{180(j-i)}{n}$ in degrees which looks prettier in radians as $\frac{(j-i)\pi}{n}$.

Therefore the answers (in degrees) are

(a) $\dfrac{180}{4} = 45°$.

(b) $\dfrac{180 \times 2}{5} = 72°$.

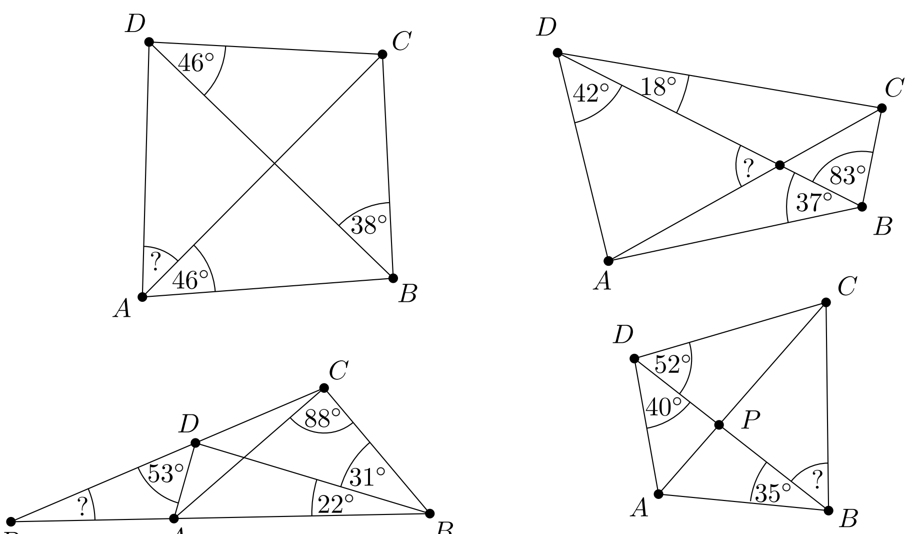
(c) $\dfrac{180 \times 3}{6} = 90°$.

(d) $\dfrac{180 \times 4}{12} = 60°$.

(e) $\dfrac{180 \times 3}{8} = 67.5°$.

(f) $\dfrac{180 \times 11}{45} = 44°$.

7. Find the angles denoted by question marks in the following diagrams. Give the explanation of why those angles are correct.



$\triangle DPC \sim \triangle APB$

Solution 7. For the top left: $ABCD$ is cyclic because $\measuredangle CDB = \measuredangle CAB$ and they open the same chord. Therefore

$$\measuredangle DAC = \measuredangle DBC = 38°.$$

For the top right: $\measuredangle CDA + \measuredangle CBA = 42 + 18 + 83 + 37 = 180$. Therefore $ABCD$ is cyclic. Then the mystery angle is

$$\frac{\overset{\frown}{AD}}{2} + \frac{\overset{\frown}{CB}}{2} = 37 + 18 = 55°.$$

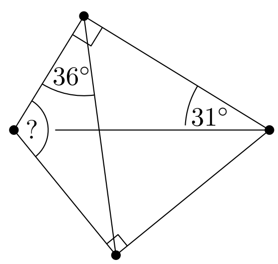
For the bottom left: $\measuredangle ADC = 180 - 53$ because CDP is a line. $\measuredangle CBA = 31 + 22 = 53$. Therefore $\measuredangle CBA + \measuredangle ADC = 180°$. Then $ABCD$ is cyclic. Therefore

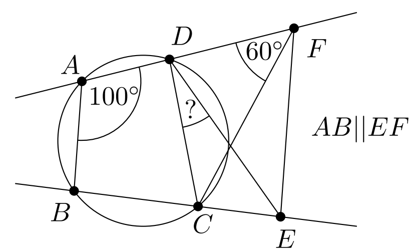
$$\begin{aligned}\measuredangle CPA &= \measuredangle CAB - \measuredangle DBA \\ &= (180 - (\measuredangle ACB + \measuredangle CBA)) - 22 \\ &= 180 - 88 - 53 - 22 \\ &= 17°.\end{aligned}$$

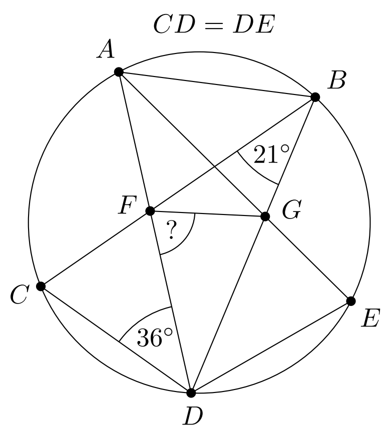
For the bottom right: Since $\triangle DPC \sim \triangle APB$, then $\measuredangle DCA = \measuredangle DBA$. Therefore $ABCD$ is cyclic. Then

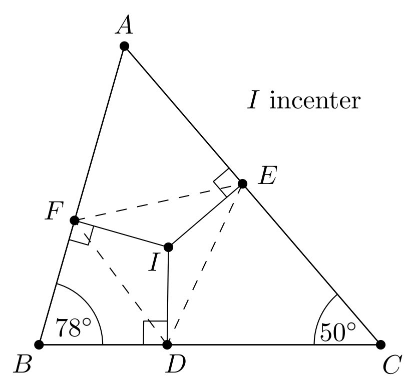
$$\measuredangle DBC = \measuredangle DAC = 180 - \measuredangle ADC - \measuredangle DCA = 180 - 40 - 52 - 35 = 53°.$$

8. Find the angles denoted by question marks in the following diagrams. Give the explanation of why those angles are correct.









Find internal angles in $\triangle DEF$

Solution 8. For the top left: Because two opposite angles are 90°, then the quadrilateral is cyclic. Suppose we label the quadrilateral $ABCD$ where A is the top vertex and we label clockwise. Then $\measuredangle ACD = 31$ because $ABCD$ is cyclic. Then

$$\measuredangle ADC = 180 - 36 - 31 = 113.$$

For the top right: From the diagram $ADCB$ is cyclic because the vertices are contained in the same circle. Then $\measuredangle DCE = 180 - \measuredangle DCB = \measuredangle DAB = 100$. Also $\measuredangle DFE = 80$ because $AB \| EF$. Then $\measuredangle DCE + \measuredangle DFE = 180$. Therefore $CEFD$ is cyclic. Then

$$\measuredangle CDE = \measuredangle CFE = 80 - 60 = 20°.$$

For the bottom left: Since $CD = ED$, then $\measuredangle EAD = \measuredangle DBC$ because they overlook chords of the same length. But that means $\measuredangle GAF = \measuredangle GBF$. Both of these angles open the chord FG. Therefore $ABGF$ is cyclic. Therefore

$$\measuredangle GFD = 180 - \measuredangle GFA = \measuredangle GBA = 21 + \measuredangle FBA = 21 + \measuredangle ADC = 21 + 36 = 57.$$

For the bottom right: Since $IE \perp AC, ID \perp BC, IF \perp AB$, then $IECD, IDBF, AFIE$ are all cyclic (they have two right angles as opposite angles). Therefore

$$\begin{aligned}
&\measuredangle EID = 180 - 50 = 130, \\
&\measuredangle FID = 180 - 78 = 102, \\
&\measuredangle FIE = 180 - \measuredangle BAC = 180 - (180 - (78 + 50)) = 78 + 50 = 128.
\end{aligned}$$

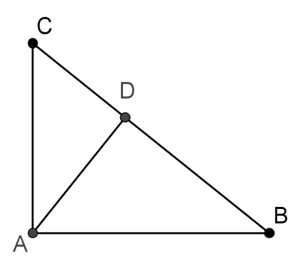
But $ID = IE = IF = r$ because I is the incenter. Therefore $\triangle IED, \triangle IEF, \triangle IDF$ are all isosceles and we can find

$$\begin{aligned}
\measuredangle IDE = \measuredangle IED &= \frac{180 - 130}{2} = 25. \\
\measuredangle IEF = \measuredangle IFE &= \frac{180 - 128}{2} = 26. \\
\measuredangle IFD = \measuredangle IDF &= \frac{180 - 102}{2} = 39.
\end{aligned}$$

Then we can conclude

$$\begin{aligned}
\measuredangle FDE &= \measuredangle IDE + \measuredangle IDF = 25 + 39 = 64. \\
\measuredangle FED &= \measuredangle IEF + \measuredangle IED = 26 + 25 = 51. \\
\measuredangle EFD &= \measuredangle IFE + \measuredangle IFD = 26 + 39 = 65.
\end{aligned}$$

BONUS Let ABC be a right triangle with $\measuredangle BAC = 90°$ satisfying that $BC = 10$ and $AD = 6$, where $AD \perp BC$ and D is in BC. Prove that no such triangle exists.



Solution 9. Let M be the midpoint of BC. Since $\triangle ABC$ is a right triangle, then $AM = BM = CM = 5$. However $AD \perp BD$ so $\triangle ADM$ is a right triangle with hypotenuse AM. Therefore $AM > AD = 6$. But that is a contradiction since $AM = 5$. Therefore this triangle does not exist.