Geometry
Homework 4 Solutions

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October 7, 2016

1. Exercises 3.3.2, 3.3.3, 3.3.4, and 3.3.5. Note that the exercises have some typos in the textbook. Here are the corrections: In 3.3.2, one of the lines should read:

$$|OP| = x_1, \quad |PQ| = \sqrt{(x_2 - x_1)^2 + y_2^2}, \quad |OQ| = \sqrt{x_2^2 + y_2^2}.$$

In 3.3.3 the equation should read:

$$(|OP| + |PQ|)^2 - |OQ|^2 = 2x_1 \left(\sqrt{(x_2 - x_1)^2 + y_2^2} - (x_2 - x_1) \right).$$

Solution 1. For 3.3.2:

$$OP = \sqrt{(x_1 - 0)^2 + (0 - 0)^2} = \sqrt{x_1^2} = |x_1| = x_1 \text{ because } x_1 > 0.$$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - 0)^2} = \sqrt{(x_2 - x_1)^2 + y_2^2}.$$

$$OQ = \sqrt{(x_2 - 0)^2 + (y_2 - 0)^2} = \sqrt{x_2^2 + y_2^2}.$$

For 3.3.3:

$$\begin{aligned}(OP + PQ)^2 - (OQ)^2 &= OP^2 + PQ^2 + 2 \cdot OP \cdot PQ - OQ^2 \\ &= x_1^2 + (x_2 - x_1)^2 + y_2^2 + 2x_1\sqrt{(x_2 - x_1)^2 + y_2^2} - (x_2^2 + y_2^2) \\ &= x_1^2 + x_2^2 - 2x_1x_2 + x_1^2 + 2x_1\sqrt{(x_2 - x_1)^2 + y_2^2} - x_2^2 \\ &= 2x_1^2 + 2x_1 \left(\sqrt{(x_2 - x_1)^2 + y_2^2} - x_2 \right) \\ &= 2x_1 \left(\sqrt{(x_2 - x_1)^2 + y_2^2} - (x_2 - x_1) \right).\end{aligned}$$

For 3.3.4: Suppose $y_2 \neq 0$. Then

$$\sqrt{(x_2 - x_1)^2 + y_2^2} > \sqrt{(x_2 - x_1)^2 + 0} = |x_2 - x_1|.$$

But then the right hand side of the equation in 3.3.3 is positive (since x_1 is positive and $|x_2 - x_1| \geq x_2 - x_1$).

For 3.3.5: If $y_2 = 0$, then the coordinate of Q is $(x_2, 0)$, so O, P, Q are all in the x-axis and hence they are collinear. That means they don't form a triangle.

2. Exercise 4.3.1.

Solution 2. Suppose the vectors $\mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ are in order. The diagonals intersect each other in the center of the square and they bisect each other. Therefore their intersection is the midpoint of the diagonal $\mathbf{v} - \mathbf{t}$. This point is the average of the two vectors, i.e., $\frac{1}{2}(\mathbf{v} + \mathbf{t})$. But the intersection is also the midpoint of the diagonal $\mathbf{w} - \mathbf{u}$ which is the average of the two vectors $\frac{1}{2}(\mathbf{w} + \mathbf{u})$. Therefore

$$\frac{1}{2}(\mathbf{v} + \mathbf{t}) = \frac{1}{2}(\mathbf{w} + \mathbf{u}).$$

That means

$$\frac{1}{4}(\mathbf{t} + \mathbf{u} + \mathbf{v} + \mathbf{w}) = \frac{1}{2}\left(\frac{\mathbf{t} + \mathbf{v}}{2} + \frac{\mathbf{u} + \mathbf{w}}{2}\right) = \frac{1}{2}(2)\left(\frac{\mathbf{t} + \mathbf{v}}{2}\right) = \frac{1}{2}(\mathbf{v} + \mathbf{t}),$$

which is the center of the square. Therefore we've proved what we set out to prove.

3. Exercises 4.3.2, 4.3.3, 4.3.4, and 4.3.5.

Solution 3. For 4.3.2: The centroid of the face opposite $\mathbf{t}$ consists of the triangle formed with the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Therefore, it's centroid is $\frac{\mathbf{u}+\mathbf{v}+\mathbf{w}}{3}$. The other faces have the following centroids:

$$\frac{\mathbf{u} + \mathbf{v} + \mathbf{t}}{3}, \quad \frac{\mathbf{u} + \mathbf{t} + \mathbf{w}}{3}, \quad \frac{\mathbf{v} + \mathbf{w} + \mathbf{t}}{3}.$$

For 4.3.3: The point 3/4 of the way from $\mathbf{t}$ to the centroid of the opposite face is

$$\frac{3}{4}\left(\frac{\mathbf{u} + \mathbf{v} + \mathbf{w}}{3} - \mathbf{t}\right) + \mathbf{t} = \frac{3}{4}\left(\frac{\mathbf{u} + \mathbf{v} + \mathbf{w}}{3}\right) + \mathbf{t} - \frac{3}{4}\mathbf{t} = \frac{\mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{t}}{4}.$$

For 4.3.4: The calculations are analogous to the one in 4.3.3.

For 4.3.5: We have that the point $\frac{1}{4}(\mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{t})$ is in the four lines connecting vertices to the centroids in the opposite face. Therefore these four lines concur.

4. Exercises 4.4.3 and 4.4.4. The equation in 4.4.4 should read as

$$\left(\mathbf{w} - \frac{\mathbf{u} + \mathbf{v}}{2}\right) \cdot (\mathbf{u} - \mathbf{v}) = 0$$

Solution 4. For 4.4.3: If $\mathbf{w}$ is equidistant from $\mathbf{u}$ and $\mathbf{v}$, then $|\mathbf{w} - \mathbf{u}| = |\mathbf{w} - \mathbf{v}|$. Therefore $|\mathbf{w} - \mathbf{u}|^2 = |\mathbf{w} - \mathbf{v}|^2$. That means

$$(\mathbf{w} - \mathbf{u}) \cdot (\mathbf{w} - \mathbf{u}) = (\mathbf{w} - \mathbf{v}) \cdot (\mathbf{w} - \mathbf{v}).$$

But then

$$\begin{aligned} \mathbf{w} \cdot \mathbf{w} - 2\mathbf{w} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} &= \mathbf{w} \cdot \mathbf{w} - 2\mathbf{w}\mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ |\mathbf{u}|^2 - 2\mathbf{w} \cdot \mathbf{u} &= |\mathbf{v}|^2 - 2\mathbf{w} \cdot \mathbf{v}. \quad (1) \end{aligned}$$

For 4.4.4: From (1) it follows that

$$\begin{aligned} |\mathbf{u}|^2 - |\mathbf{v}|^2 - (2\mathbf{w} \cdot \mathbf{u} - 2\mathbf{w} \cdot \mathbf{v}) &= 0 \\ (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) - 2\mathbf{w} \cdot (\mathbf{u} - \mathbf{v}) &= 0 \\ (\mathbf{u} + \mathbf{v} - 2\mathbf{w}) \cdot (\mathbf{u} - \mathbf{v}) &= 0. \end{aligned}$$

Since we can divide by a scalar, if we divide by -2, we get

$$\left(\mathbf{w} - \frac{\mathbf{u} + \mathbf{v}}{2}\right) \cdot (\mathbf{u} - \mathbf{v}) = 0.$$

The term $\mathbf{w} - \frac{\mathbf{u}+\mathbf{v}}{2}$ is a vector from the midpoint of $\mathbf{u}$ and $\mathbf{v}$ to the vector $\mathbf{w}$. The equation says that this vector is perpendicular to the vector $\mathbf{u} - \mathbf{v}$ (which describes the line from $\mathbf{v}$ to $\mathbf{u}$). Therefore $\mathbf{w}$ has to be in the perpendicular line bisector of $\mathbf{u}$ and $\mathbf{v}$.

5. Exercises 4.5.2 and 4.5.3.

Solution 5. For 4.5.2:

$$(\mathbf{v}+\mathbf{u})\cdot(\mathbf{v}-\mathbf{u}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{u} = |\mathbf{v}|^2 - |\mathbf{u}|^2.$$

For 4.5.3: The vector from $-\mathbf{u}$ to $\mathbf{v}$ is $\mathbf{v}-(-\mathbf{u}) = \mathbf{v}+\mathbf{u}$. The vector from $\mathbf{u}$ to $\mathbf{v}$ is $\mathbf{v}-\mathbf{u}$. Then the dot product of those vectors, by 4.5.2, is $|\mathbf{v}|^2 - |\mathbf{u}|^2$. But, $\mathbf{u}$ and $\mathbf{v}$ are vectors in the same circle, so $|\mathbf{u}| = |\mathbf{v}|$. Therefore $|\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$. Therefore the dot product is zero and hence the vectors are perpendicular.

6. Exercises 4.6.2, 4.6.3, and 4.6.4.

Solution 6. For 4.6.2:

$$\begin{aligned}(\mathbf{u}+x\mathbf{v})\cdot(\mathbf{u}+x\mathbf{v}) &= \mathbf{u}\cdot u + 2x(\mathbf{u}\cdot\mathbf{v}) + x^2\mathbf{v}\cdot\mathbf{v} \\ &= |\mathbf{u}|^2 + 2x(\mathbf{u}\cdot\mathbf{v}) + x^2|\mathbf{v}|^2. \qquad (2)\end{aligned}$$

The left hand side of (2) is the square of the length of a vector and therefore nonnegative. That means that

$$|\mathbf{u}|^2 + 2x(\mathbf{u}\cdot\mathbf{v}) + x^2|\mathbf{v}|^2 \geq 0 \quad \text{for any real number } x. \qquad (3)$$

For 4.6.3: Suppose $Ax^2+Bx+C \geq 0$ for all $x \geq 0$. Let $f(x) = Ax^2+Bx+C$. Since $f(x)$ is never negative, $f(x) = 0$ has at most one solution. Indeed, if $f(x) = 0$ for two different values of x called x_1 and x_2 with $x_1 < x_2$, then $f(x)$ (which graphs as a parabola) would cross the x-axis into the negative side in either $(-\infty, x_1), (x_1, x_2)$, or (x_2, ∞). In summary $f(x) = 0$ has at most one solution. However, we know by the quadratic formula, that $f(x) = 0$ has the following solutions:

$$x = \frac{-B \pm \sqrt{B^2-4AC}}{2A}.$$

The only way this does not describe two solutions is because $B^2 - 4AC = 0$ or because $\sqrt{B^2-4AC}$ does not exist, which implies that $B^2 - 4AC < 0$. Therefore

$$B^2 - 4AC \leq 0. \qquad (4)$$

For 4.6.4: In (3), let $f(x) = |\mathbf{v}|^2x^2 + 2(\mathbf{u}\cdot\mathbf{v})x + |\mathbf{u}|^2$. Let $A = |\mathbf{v}|^2$, $B = 2(\mathbf{u}\cdot\mathbf{v})$, and $C = |\mathbf{u}|^2$. Then by applying (4), we get

$$\begin{aligned}4(\mathbf{u}\cdot\mathbf{v})^2 - 4|\mathbf{v}|^2|\mathbf{u}|^2 &\leq 0 \\ (\mathbf{u}\cdot\mathbf{v})^2 &\leq |\mathbf{u}|^2|\mathbf{v}|^2 \\ |\mathbf{u}\cdot\mathbf{v}| &\leq |\mathbf{u}||\mathbf{v}|.\end{aligned}$$

Remark 1. The proof also gives us information of when we have equality in Cauchy-Schwarz. We have $f(x) \geq 0$ for all x. From the work, we see that the only way we get equality is if $B^2 - 4AC = 0$. But that means that $f(x) = 0$ has a solution. But

$$f(x) = (\mathbf{u}+x\mathbf{v})\cdot(\mathbf{u}+x\mathbf{v}) = |\mathbf{u}+x\mathbf{v}|^2.$$

That is, $f(x)$ is the square of the length of a vector. The only way it can be zero, is if the vector itself is zero. Therefore

$$\mathbf{u}+x\mathbf{v} = 0 \quad \Rightarrow \quad \mathbf{u} = -x\mathbf{v} \quad \text{that is } \mathbf{u} \text{ is a scalar multiple of } \mathbf{v}.$$

7. Let P be a point inside square $ABCD$ such that $PA = 2$, $PB = 3$, $PC = 4$. Compute PD.

Solution 7. Let A be in the origin, B be at $(s,0)$, $D=(0,s)$ and $C=(s,s)$, where s is the side length of the square. Let $P=(x,y)$. Then we have

$$\begin{aligned} PA^2 &= x^2+y^2 = 4 \\ PB^2 &= (x-s)^2+y^2 = 9 \\ PC^2 &= (x-s)^2+(y-s)^2 = 16. \end{aligned}$$

Note

$$PD^2 = x^2+(y-s)^2 = (x^2+y^2)+((x-s)^2+(y-s)^2)-((x-s)^2+y^2) = 4+16-9 = 11.$$

Therefore $PD=\sqrt{11}$.

8. Let $ABCD$ be a rhombus with a point P on the side BC and Q on the side CD such that $BP=CQ$. Prove that the centroid of the triangle APQ lies on the segment BD.

Solution 8. Move the rhombus so that its diagonals intersect at the origin. Since it's a rhombus, the diagonals bisect each other so $A+C=B+D=0$. Since P is on BC, then $P=B+(C-B)t$ for some nonnegative real number t. Since Q is in CD then $Q=C+(D-C)s$ for some nonnegative real number s. We know that $|BP|=|CQ|$, but

$$|BP| = |P-B| = |(C-B)t| = |C-B|t,$$

and

$$|CQ| = |Q-C| = |(D-C)s| = |D-C|s.$$

Since it's a rhombus we have $|C-B|=|D-C|$. Since $|BP|=|CP|$, we can conclude that $t=s$. Then the centroid of APQ is

$$\begin{aligned} \frac{A+P+Q}{3} = \frac{A+B+(C-B)t+C+(D-C)t}{3} &= \frac{(A+C)+B+(D-B)t}{3} \\ &= B+\frac{-2B+(D-B)t}{3} \\ &= B+\frac{2D+(D-B)t}{3} \\ &= B+\left(\frac{t+1}{3}\right)(D-B). \end{aligned}$$

For the last equality we used that $2D=D-B$. Since the centroid of APQ is of the form $B+\alpha(D-B)$, then it's on the line BD.

BONUS Let $\triangle ABC$ be an equilateral triangle. Suppose P is a point inside the triangle satisfying that $AP=3, BP=4$, and $CP=5$. Find the length of the side of equilateral triangle, i.e., find AB.

Solution 9. Suppose the equilateral triangle has side ℓ. Now, the height of the equilateral triangle is $\sqrt{3}2\ell$. We can place the equilateral triangle in the Cartesian plane in a way that the midpoint of a side length is in the origin. Then the coordinates are $A=(-\ell/2,0), B=(\ell/2,0), C=(0,\sqrt{3}/2\ell)$. Let $P=(x,y)$. Therefore

$$\begin{aligned} AP^2 &= \left(x+\frac{\ell}{2}\right)^2+y^2 = x^2+\ell x+\frac{\ell^2}{4}+y^2 = 9 \\ BP^2 &= \left(x-\frac{\ell}{2}\right)^2+y^2 = x^2-\ell x+\frac{\ell^2}{4}+y^2 = 16 \\ CP^2 &= x^2+\left(y-\frac{\sqrt{3}\ell}{2}\right)^2 = x^2+y^2-\sqrt{3}\ell y+\frac{3\ell^2}{4} = 25. \end{aligned}$$

Then, if we subtract $AP^2 - BP^2$, we get

$$-7 = AP^2 - BP^2 = 2\ell x \quad \Rightarrow 4\ell^2 x^2 = (-7)^2 = 49 \quad \Rightarrow 12\ell^2 y^2 = 147. \tag{5}$$

If we instead add $AP^2 + BP^2$, we get

$$25 = 2(x^2 + y^2) + \frac{\ell^2}{2} \quad \Rightarrow x^2 + y^2 = \frac{25}{2} - \frac{\ell^2}{4}. \tag{6}$$

Plugging this last equation into the equation of CP^2 we get

$$\begin{aligned} 25 &= \frac{25}{2} - \frac{\ell^2}{4} - \sqrt{3}\ell y + \frac{3\ell^2}{4} \\ \frac{25}{2} &= -\sqrt{3}\ell y + \frac{\ell^2}{2} \\ 2\sqrt{3}\ell y &= -25 + \ell^2 \\ 12\ell^2 y^2 &= (-25 + \ell^2)^2 = 625 - 50\ell^2 + \ell^4. \end{aligned} \tag{7}$$

Adding (5) and (7) we get

$$12\ell^2(x^2 + y^2) = 147 + 625 - 50\ell^2 + \ell^4 = 772 - 50\ell^2 + \ell^4.$$

Plugging in (6) we get

$$12\ell^2(x^2 + y^2) = 12\ell^2 \left(\frac{25}{2} - \frac{\ell^2}{4}\right) = \ell^2 \left(150 - 3\ell^2\right) = 150\ell^2 - 3\ell^4.$$

Therefore

$$\begin{aligned} 150\ell^2 - 3\ell^4 &= 772 - 50\ell^2 + \ell^4 \\ 4\ell^4 - 200\ell^2 + 772 &= 0 \\ \ell^4 - 50\ell^2 + 193 &= 0. \end{aligned}$$

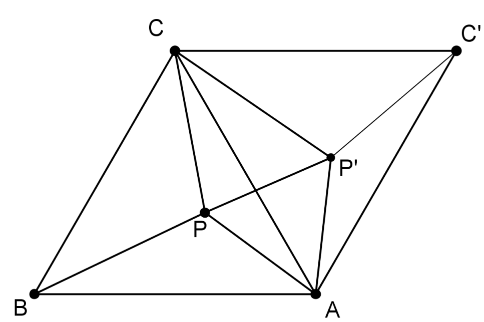
Therefore

$$\ell^2 = \frac{50 \pm \sqrt{50^2 - 4 \cdot 193}}{2} = 25 \pm \sqrt{25^2 - 193} = 25 \pm \sqrt{432} = 25 \pm 12\sqrt{3}.$$

Since $\sqrt{25 - 12\sqrt{3}} < 3$, then P would be outside the triangle in that case. Therefore

$$\ell = \sqrt{25 + 12\sqrt{3}}.$$

Alternative Solution: Rotate the figure 60 degrees clockwise with center at A. Then B goes to C and C goes to a new point C' while A stays put. Also, P goes to P' as in the figure.



Then by construction $BP = CP'$, $AP' = AP$ and $C'P' = CP$. Since $AP' = AP$ and $\measuredangle PAP' = 60°$, then $\triangle APP'$ is equilateral. Therefore $PP' = AP$. Then the triangle $CP'P$ has side lengths $CP' = BP = 4, PP' = AP = 3$, and $CP = 5$. Since $3^2 + 4^2 = 5^2$, then $\triangle CP'P$ is a right triangle. But then

$$\measuredangle CP'A = \measuredangle CP'P + \measuredangle PP'A = 90° + 60° = 150°.$$

Then by Law of Cosines we have

$$\ell^2 = AC^2 = CP'^2 + AP'^2 - 2AP'CP' \cos(150°) = 4^2 + 3^2 - 2(3)(4)\left(-\frac{\sqrt{3}}{2}\right) = 25 + 12\sqrt{3}.$$

Therefore $\ell = \sqrt{25 + 12\sqrt{3}}$.