Geometry
Homework 5 Solutions

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1. Exercises 5.1.1, 5.1.2, and 5.1.3.

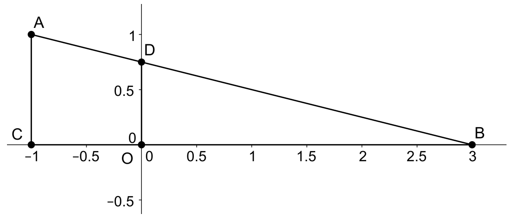
   Solution 1.
   For **5.1.1**: The equation of the line from $(-1,1)$ to $(n,0)$ is

$$y = \frac{1}{-1-n}(x-n).$$

It crosses the y-axis when $x=0$. Therefore

$$y = -\frac{1}{n+1}(-n) = \frac{n}{n+1}.$$

Alternative Solution: Alternatively, you could use Thales theorem. Label the points as in the following figure:



Then, what we want to show is that $OD = \frac{n}{n+1}$. Since $OD \| AC$, then by Thales Theorem

$$\frac{OD}{AC} = \frac{OB}{BC}.$$

Since $AC = 1$, $OB = n$, and $BC = OB + OC = n+1$, then $OD = \frac{n}{n+1}$.
For **5.1.2**: The n-th point has the form $\frac{n}{n+1}$ (where we start with 0). Now

$$f(y) = f\left(\frac{n}{n+1}\right) = \frac{1}{2-\left(\frac{n}{n+1}\right)} = \frac{1}{\left(\frac{2(n+1)-n}{n+1}\right)} = \frac{n+1}{2n+2-n} = \frac{n+1}{n+2}.$$

But that's the next point in the sequence. Therefore $f(y)$ does indeed make the claimed move.
For **5.1.3**:

$$\begin{aligned} f(y) &= y \\ \frac{1}{2-y} &= y \\ y(2-y) &= 1 \\ y^2 - 2y + 1 &= 0 \\ y &= 1. \end{aligned}$$

Therefore the point where y is not moved is $y = 1$. This is the spot where $AD \| BC$. The significance is that it shows the place where the "point of infinity" is projected.

2. Exercises 5.3.1 and 5.3.2.

Solution 2.
For **5.3.1**: We have

$$0a + 0b + c = 0$$
$$a + b + c = 0.$$

Therefore $c = 0$. We can fix $a = 1$, then $b = -1$. The plane looks like $x - y = 0$. This plane does not go through $(1, 0, 0)$ or $(0, 1, 0)$ because $1 - 0 \neq 0$ and $0 - 1 \neq 0$.

For **5.3.2**: By symmetry, the points $(0, 0, 1), (0, 1, 0)$, and $(1, 0, 0)$ behave similarly. So by considering the case above and the case in class (the plane through the origin containing $(1, 0, 0)$ and $(0, 1, 0)$ does not go through $(0, 0, 1)$ or $(1, 1, 1)$), we have considered all possible ways where three points could have a line in common. Therefore the points $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$ represent 4 "points" such that no 3 of them are in a "line".

Now the exercise itself talks about four "lines" no three of which have a "point" in common which is not exactly the same thing as above. However, by thinking of the points as representing the coefficients of the planes through the origin (for example $(1, 1, 1)$ represents the plane $x + y + z = 0$) and by thinking of the intersection of two planes by solving for (x, y, z), the work above is identical but switching the roles of (a, b, c) with those of (x, y, z). Therefore the same example that works for "4 'points' no three of which are in a 'line' ", works for "4 'lines' no three of which have a 'point' in common".

3. Exercises 5.3.3 and 5.3.4.

Solution 3.
For **5.3.3**: The "line" through AB is a plane containing O, A, B. A "line" through BC is a plane containing O, B, C. The intersection of those two planes is a line and it contains O and B, therefore it is the line OB. Therefore $E = B$.

For **5.3.4**: Suppose that AB, BC, CD have a common "point". Since the common "point" of AB and BC is B, and the common "point" between BC and CD is C, then $B = C$. But B and C are supposed to be different "points". So we have a contradiction.

The other cases are (AB, BC, AD), (AB, CD, AD), (BC, CD, AD). In the first one we have $B = A$, in the second $A = D$, and in the third $C = D$. In all cases two points are forced to be equal.

4. Exercise 5.4.1.

Solution 4. We have

$$a + 2b + 3c = 0$$
$$a + b + c = 0.$$

Then $b + 2c = 0$. We can let $c = 1$, so $b = -2$ and $a = -b - c = 2 - 1 = 1$. Therefore the plane $x - 2y + z = 0$ contains $(1, 2, 3)$ and $(1, 1, 1)$.

5. Exercises 5.4.2 and 5.4.3.

Solution 5.
For **5.4.2**: The line of intersection is the line that goes from the origin to $(1, -2, 1)$.

For **5.4.3**: The reason we could use the answer from 5.4.1 is that in homogeneous equations, we just change $(a, b, c) \leftrightarrow (x, y, z)$.

6. Exercises 5.5.1 and 5.5.2.

Solution 6.
For **5.5.1**:

$$f_1(f_2(x)) = f_1(a_2x + b_2) = a_1(a_2x + b_2) + b_1 = a_1a_2x + (a_1b_2 + b_1) = Ax + B,$$

where $A = a_1a_2$ and $B = a_1b_2 + b_1$. Since $a_1 \neq 0$ and $a_2 \neq 0$, then $A = a_1a_2 \neq 0$.

For **5.5.2**: Sending $x \to x + \ell$ is a function of the form $f(x) = ax + b$ where $a = 1$ and $b = \ell$. Sending $x \to kx$ for $k \neq 0$ is a function of the form $f(x) = ax + b$ where $a = k \neq 0$ and $b = 0$. Therefore it has the shape of the functions f_1, f_2 above. If we compose them, we get another function of the form $Ax + B$ (with $A \neq 0$), which is also of the same form f_1, f_2, so we can keep composing them and end with something of the form $f(x) = ax + b$ with $a \neq 0$.

7. Exercise 5.5.3.

Solution 7. Consider parallel lines $\mathfrak{L}_1$, $\mathfrak{L}_2$. Label numbers in each line in such a way that they are equally spaced and the zeroes are aligned. Let the point 0 in $\mathfrak{L}_1$ be called A and 0 in $\mathfrak{L}_2$ be called B . If the line PA is perpendicular to $\mathfrak{L}_1$, then we get $f(x) = kx$ for a nonzero k as shown in Figure 5.14. If PA is not perpendicular, then let A' be the intersection of PA with $\mathfrak{L}_2$. Let $BA' = b$. Then $f(0) = b$. Now consider the point X in $\mathfrak{L}_1$ that is x units away from A, and say it maps to X' in $\mathfrak{L}_2$ which is $f(x)$ units away from B. By Thales theorem we have that $AX/A'X' = PA/PA' = k$, for a fixed number $k \neq 0$. But then

$$A'X' = AX \times \left(\frac{PA'}{PA}\right) = x\left(\frac{1}{k}\right) = ax,$$

for $a = 1/k \neq 0$. Therefore

$$f(x) = BX' = A'X' + A'B = ax + b,$$

with $a \neq 0$.

8. Exercises 5.6.1 and 5.6.2.

Solution 8.
For **5.6.1**:

$$\begin{aligned} y &= \frac{ax+b}{cx+d} \\ (cx+d)y &= ax+b \\ (cy-a)x &= b-dy \\ x &= \frac{b-dy}{cy-a}. \end{aligned}$$

The last step requires $cy - a \neq 0$. Let's show that $ad - bc \neq 0$ implies $cy - a \neq 0$. This is equivalent to showing that $cy - a = 0$ implies $ad - bc = 0$. Suppose $cy - a = 0$. Then we have $b - dy = 0$ to make $(cy - a)x = b - dy$ true. Therefore $a = cy$ and $b = dy$. So $ad = cdy$ and $bc = cdy$, so $ad - bc = 0$. Therefore when we divided by $cy - a$ we were assuming that $ad - bc \neq 0$.

For **5.6.2**:

$$\begin{aligned} f_1(f_2(x)) &= f_1\left(\frac{a_2x+b_2}{c_2x+d_2}\right) \\ &= \frac{a_1\left(\frac{a_2x+b_2}{c_2x+d_2}\right)+b_1}{c_1\left(\frac{a_2x+b_2}{c_2x+d_2}\right)+d_1} \\ &= \frac{\left(\frac{a_1a_2x+a_1b_2+b_1c_2x+b_1d_2}{c_x+d_2}\right)}{\left(\frac{a_2c_1x+b_2c_1+c_2d_1x+d_1d_2}{c_2x+d_2}\right)} \\ &= \frac{(a_1a_2+b_1c_2)x+(a_1b_2+b_1d_2)}{(a_2c_1+c_2d_1)x+(b_2c_1+d_1d_2)} = \frac{Ax+B}{Cx+D}. \end{aligned}$$