Geometry
Homework 6 Solutions

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1. Exercises 5.7.1 and 5.7.2.

   Solution 1.
   For **5.7.1**: Let $[p, q; r, s] = K$, where p, q, r, K are fixed. Let's show there's a unique s that satisfies this.

$$\begin{aligned} K &= [p, q; r, s] \\ K &= \frac{(r-p)(s-q)}{(r-q)(s-p)} \\ (s-p)(r-q)K &= (r-p)(s-q) \\ (sr - sq)K + (pq - pr)K &= rs - ps + pq - rq \\ s((r-q)K + (p-r)) &= q(p-r) + p(r-q)K \\ s &= \frac{q(p-r) + p(r-q)K}{(r-q)K + (p-r)}. \end{aligned}$$

   Therefore s has only one option. We work with the understanding that if $(r-q)K + (r-p) = 0$, then $s = \infty$ and if $(r-q)K + (r-p) = \infty$, then $s = 0$.
   For **5.7.2**: $p = 0, q = 2, r = 3, K = 4/3$. Then

$$s = \frac{q(p-r) + p(r-q)K}{(r-q)K + (p-r)} = \frac{2(0-3) + 0(3-2)(4/3)}{(3-2)(4/3) + (0-3)} = \frac{-6}{4/3 - 3} = \frac{18}{5} = 3.6.$$

   In Figure 5.18, the first line has $p = 0, q = 2, r = 3, s = 3.5$. The second one has $p = 0, q = 2, r = 3, s \approx 3.6$, the third one looks like $p = 0, q = 2, r = 3, s \approx 3.7$. Therefore the correct answer is the second one.

2. Exercises 5.7.3.

   Solution 2. To have distances 1, e, e^2, the points should be $p = 0, q = 1, r = 1 + e, s = 1 + e + e^2$. Then

$$[p, q; r, s] = \frac{(r-p)(s-q)}{(r-q)(s-p)} = \frac{(1+e)(e+e^2)}{e(1+e+e^2)} = \frac{e(1+e)^2}{e(1+e+e^2)} = \frac{1+2e+e^2}{1+e+e^2} = 1 + \frac{e}{1+e+e^2}.$$

   For this to mimic equally spaced, it would need to equal 4/3. But that means

$$\begin{aligned} \frac{e}{1+e+e^2} &= \frac{1}{3} \\ 3e &= 1 + e + e^2 \\ e^2 - 2e + 1 &= 0 \\ (e-1)^2 &= 0 \\ e &= 1 \end{aligned}$$

The only way this works is if $e = 1$, but in that case, you have the normal Cartesian plane and not a projective plane. So it doesn't quite work for the purposes of perspective.

3. Exercises 5.8.1 and 5.8.2.

Solution 3.
For **5.8.1**:

$$[p, q; s, r] = \frac{(s-p)(r-q)}{(s-q)(r-p)} = \left(\frac{(r-p)(s-q)}{(r-q)(s-p)}\right)^{-1} = \frac{1}{y}.$$

For **5.8.2**:

$$[q, p; r, s] = \frac{(r-q)(s-p)}{(r-p)(s-q)} = \left(\frac{(r-p)(s-q)}{(r-q)(s-p)}\right)^{-1} = \frac{1}{y}.$$

4. Exercise 5.8.3.

Solution 4.

$$\begin{aligned}[p, q; r, s] + [p, r; q, s] &= \frac{(r-p)(s-q)}{(r-q)(s-p)} + \frac{(q-p)(s-r)}{(q-r)(s-p)} = \frac{(r-p)(s-q) - (q-p)(s-r)}{(r-q)(s-p)} \\ &= \frac{rs - rq - ps + pq - qs + qr + ps - pr}{(r-q)(s-p)} = \frac{rs + pq - qs - pr}{(r-q)(s-p)} \\ &= \frac{r(s-p) - q(s-p)}{(r-q)(s-p)} = \frac{(r-q)(s-p)}{(r-q)(s-p)} = 1.\end{aligned}$$

5. Exercises 5.8.4, 5.8.5 and 5.8.6.

Solution 5.
For **5.8.4**: We will apply each of the three functions to $y, 1/y, 1-y$ in that order.

Applying y to any of the functions yields $y, 1/y, 1-y$.

Applying $1/y$ to any of the functions changes them to $1/y, y, \frac{1}{1-y}$.

Applying $1-y$ to any of the functions yields $1-y, 1-\frac{1}{y}, 1-(1-y) = y$.

But now we have 5 functions: $y, 1/y, 1-y, 1-\frac{1}{y}, \frac{1}{1-y}$. We apply each of the three functions to the "new guys" $\frac{1}{1-y}$ and $1-\frac{1}{y}$.

$y \to \frac{1}{1-y}, 1-\frac{1}{y}$.

$\frac{1}{y} \to 1-y, \frac{y}{y-1}$.

$1-y \to 1-\frac{1}{1-y} = \frac{1-y-1}{1-y} = \frac{-y}{1-y} = \frac{y}{y-1}, 1-\left(1-\frac{1}{y}\right) = \frac{1}{y}$.

There is only one newcomer: $\frac{y}{y-1}$. If we apply the three functions to this one we get $\frac{y}{y-1}, \frac{y-1}{y} = 1-\frac{1}{y}, 1-\frac{y}{y-1} = \frac{1}{y}$. So there is no new function that is revealed. So any further composition will remain in one of these six functions.

For **5.8.5**: Suppose we have a permutation of $\{p, q, r, s\}$. There are 24 possibilities. For each of the possibilities, let's show that you can get to them by doing a series of switches involving only the first two, the middle two, or the last two numbers.

$\{p, q, r, s\}$: No change necessary.

$\{p, q, s, r\}$: Switch the last two.

$\{p, r, q, s\}$: Switch the middle two.

$\{p, r, s, q\}$: Switch the middle two, then switch the last two.

$\{p, s, q, r\}$: Switch the middle two, then switch the last two.

$\{p, s, r, q\}$: Switch the last two, then switch the middle two, then switch the last two.

$\{q, p, r, s\}$: Switch the first two.

$\{q, p, s, r\}$: Switch the first two, then switch the last two.

$\{q, r, p, s\}$: Switch the first two, then switch the middle two.

$\{q, r, s, p\}$: Switch the first two, then the middle two, then the last two.

$\{q, s, p, r\}$: Switch the middle two, then switch the first two, then switch the last two.

$\{q, s, r, p\}$: Switch the last two, then you are in the previous situation.

$\{r, q, p, s\}$: Switch the first two, then the middle two, then the first two again.

$\{r, q, s, p\}$: Switch the first two, then the middle two, then the first two again, then the last two.

$\{r, p, q, s\}$: Switch the first two, then the middle two, then the first two, then the middle two.

$\{r, p, s, q\}$: Switch the first two, then the middle two, then the first two, then the middle two, then the last two.

$\{r, s, p, q\}$: Switch the middle two and you get to the previous situation.

$\{r, s, q, p\}$: Switch the last two and you get to the previous situation.

$\{s, q, r, p\}$: Switch the first two, then the middle two, then the last two, then the middle two, then the first two.

$\{s, q, p, r\}$: Switch the first two, then the middle two, then the last two, then the middle two, then the first two, then the last two.

$\{s, r, q, p\}$: Switch the first two, then the middle two, then the last two, then the middle two, then the first two, then the middle two.

$\{s, r, p, q\}$: Switch the first two, then the middle two, then the last two, then the middle two, then the first two, then the middle two, then the last two.

$\{s, p, q, r\}$: Switch the first two, then switch the middle two, then switch the last two.

$\{s, p, r, q\}$: Switch the last two and you get to the previous situation.

For **5.8.6**: Swapping the first two or the last two yields $y \to 1/y$. Swapping the middle two yields $1-y$, and not doing a change means $y \to y$. Since every permutation comes from one of these transformations and the composition of these transformations has to be of the form $y, \frac{1}{y}, 1-y, \frac{1}{1-y}, \frac{y}{y-1}, 1-\frac{1}{y}$.

Remark 1. Exercise 5.8.5 can be recast in the language of permutation groups. The exercise asks one to prove that every permutation of 4 elements $\{1, 2, 3, 4\}$ can be written as a product of the transpositions $(12), (23), (34)$. First note than every transposition is a product of transpositions of the form $(12), (23), (34)$.

$$(13) = (12)(23)(12), \qquad (14) = (12)(23)(34)(23)(12), \qquad (24) = (23)(34)(23).$$

Since every permutation can be written as a product of transpositions and every transposition can be written as a product of transpositions of the form $(12), (23), (34)$, then the statement follows.

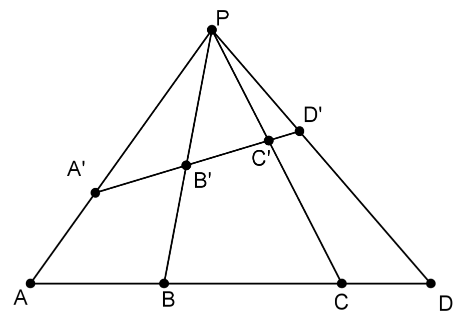
6. We can define the cross-ratio in the plane as follows. Given four points A, B, C, D on a line, then we say that the cross-ratio $[A, B; C, D]$ is

$$[A, B; C, D] = \left(\frac{AC}{BC}\right) / \left(\frac{AD}{BD}\right).$$

In the following diagram, prove that $[A, B; C, D] = [A', B'; C', D']$ (true whenever AA', BB', CC', and DD' concur):

Solution 6. Let $\measuredangle APB = \alpha, \measuredangle BPC = \beta, \measuredangle CPD = \gamma, \measuredangle ABP = \delta, \measuredangle A'B'P = \delta', \measuredangle ACP = \epsilon$, $\measuredangle A'C'P = \epsilon', \measuredangle ADP = \theta \measuredangle AD'P' = \theta'$. Then by Law of Sines on $\triangle ACP$ and on $\triangle A'C'P$ we have

$$\frac{AC}{\sin(\alpha+\beta)} = \frac{AP}{\sin(\epsilon)} \qquad\qquad \frac{A'C'}{\sin(\alpha+\beta)} = \frac{A'P}{\sin(\epsilon')}.$$



Similarly using $\triangle BCP, \triangle B'C'P$ we have

$$\frac{BC}{\sin(\beta)} = \frac{BP}{\sin(\epsilon)} \qquad\qquad \frac{B'C'}{\sin(\beta)} = \frac{B'P}{\sin(\epsilon')}.$$

Using $\triangle ADP, \triangle A'D'P$ we have

$$\frac{AD}{\sin(\alpha+\beta+\gamma)} = \frac{AP}{\sin(\theta)} \qquad\qquad \frac{A'D'}{\sin(\alpha+\beta+\gamma)} = \frac{A'P}{\sin(\theta')}.$$

Using $\triangle BDP, \triangle B'D'P$ we have

$$\frac{BD}{\sin(\beta+\gamma)} = \frac{BP}{\sin(\theta)} \qquad\qquad \frac{B'D'}{\sin(\beta+\gamma)} = \frac{B'P}{\sin(\theta')}.$$

Then combining all of this we get

$$[A,B;C,D] = \frac{\left(\frac{AC}{BC}\right)}{\left(\frac{AD}{BD}\right)} = \frac{\left(\frac{\left(\frac{AP\sin(\alpha+\beta)}{\sin(\epsilon)}\right)}{\left(\frac{BP\sin(\beta)}{\sin(\epsilon)}\right)}\right)}{\left(\frac{\left(\frac{AP\sin(\alpha+\beta+\gamma)}{\sin(\theta)}\right)}{\left(\frac{BP\sin(\beta+\gamma)}{\sin(\theta)}\right)}\right)} = \frac{\left(\frac{AP}{BP}\right)\left(\frac{\sin(\alpha+\beta)}{\sin(\beta)}\right)}{\left(\frac{AP}{BP}\right)\left(\frac{\sin(\alpha+\beta+\gamma)}{\sin(\theta)}\right)} = \frac{\left(\frac{\sin(\alpha+\beta)}{\sin(\beta)}\right)}{\left(\frac{\sin(\alpha+\beta+\gamma)}{\sin(\theta)}\right)}.$$

and

$$[A',B';C',D'] = \frac{\left(\frac{A'C'}{B'C'}\right)}{\left(\frac{A'D'}{B'D'}\right)} = \frac{\left(\frac{\left(\frac{A'P\sin(\alpha+\beta)}{\sin(\epsilon')}\right)}{\left(\frac{B'P\sin(\beta)}{\sin(\epsilon')}\right)}\right)}{\left(\frac{\left(\frac{A'P\sin(\alpha+\beta+\gamma)}{\sin(\theta')}\right)}{\left(\frac{B'P\sin(\beta+\gamma)}{\sin(\theta')}\right)}\right)} = \frac{\left(\frac{A'P}{B'P}\right)\left(\frac{\sin(\alpha+\beta)}{\sin(\beta)}\right)}{\left(\frac{A'P}{B'P}\right)\left(\frac{\sin(\alpha+\beta+\gamma)}{\sin(\theta)}\right)} = \frac{\left(\frac{\sin(\alpha+\beta)}{\sin(\beta)}\right)}{\left(\frac{\sin(\alpha+\beta+\gamma)}{\sin(\theta)}\right)}.$$

Therefore $[A,B;C,D] = [A',B';C',D']$.

Remark 2. In class we proved that if four projective points project to another, then their cross-ratios are equal. If we think of the points in terms of vectors, then if $p = A$, $q = B$, $r = C$, $s = D$, we have $r - p = \bar{AC}, r - q = \bar{BC}, s - p = \bar{AD}, s - q = \bar{BD}$, so

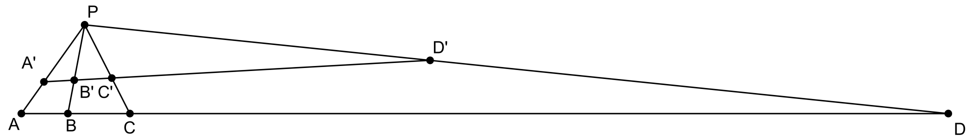
$$[p,q;r,s] = \frac{(r-p)(s-q)}{(r-q)(s-p)} = \frac{AC \times BD}{BC \times AD} = [A,B;C,D].$$

Since A, B, C, D project onto A', B', C', D', then their cross rations must be equal to each other.

7. Suppose we have A, B, C three points aligned. Let D be the point at infinity. Then show

$$[A,B;C,D] = \frac{AC}{BC}.$$

In other words, as D goes farther and farther away, the cross ratio approaches AC/BC.



Solution 7. Let D be a point far away and consider it as a variable. Then

$$[A, B; C, D] = \left(\frac{AC}{BC}\right)\left(\frac{BD}{AD}\right).$$

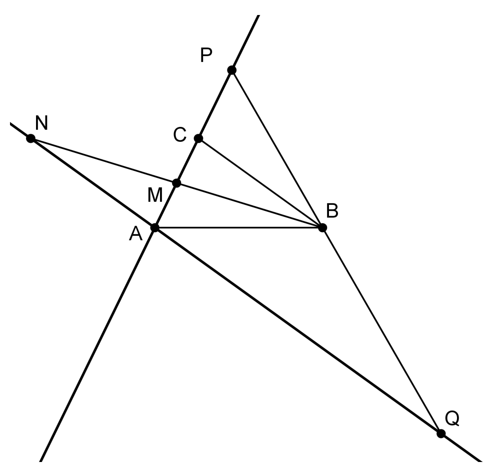
AS $D \to \infty$, then $\frac{BD}{AD} = 1$. Therefore

$$[A, B; C, D] = \frac{AC}{BC}.$$

8. Let ABC be a triangle, let M be the midpoint of AC, and let N be a point on the line BM such that AN is parallel to BC. Let P be any point on the line AC, and let Q be the intersection of the line BP with the line AN. Prove that

$$\frac{AQ}{QN} = \frac{1}{2}\left(\frac{AP}{PM}\right).$$

Hint: Use cross ratios.



Solution 8.

$$\frac{AQ}{QN} = [A, N; Q, \infty]$$

and

$$\frac{1}{2}\left(\frac{AP}{PM}\right) = \left(\frac{MC}{CA}\right)\left(\frac{AP}{PM}\right) = [A, M; P, C].$$

Consider the point of perspective to be B. Project the line AP to the line QN. The projection of M is N, the projection of A is A, the projection of C is ∞ (because $BC \| AN$), and the projection of P is Q. Therefore

$$[A, M; P, C] = [A, N; Q, \infty],$$

and hence the problem is solved.