Geometry
Homework 7 Solutions

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1. Exercises 6.1.1 and 6.1.4.

   Solution 1.
   For **6.1.1**: Suppose that the "line" starts with a line of slope k (followed by a line of slope $k/2$). Since the line has slope k at first, the equation of the line is $y = kx - 1$ (since the y-intercept is -1). This line meets the x-axis at $(1/k, 0)$ (since $kx - 1 = 0$, has solution $x = 1/k$). From that point to $(2, 1/2)$, the slope should be $k/2$, so we have

$$\begin{aligned}\frac{\frac{1}{2}}{2-\frac{1}{k}} &= \frac{k}{2}\\ 1 &= k\left(2-\frac{1}{k}\right)\\ 1 &= 2k-1\\ 1 &= k.\end{aligned}$$

   Therefore, the "line" meets the x-axis at $x = 1$.

   For **6.1.4**: A lot of points work. For example $(0, 0), (2, 1/2), (0, -1), (2, -1)$ works.

2. Exercises 6.1.2 and 6.1.3.

   Solution 2.
   For **6.1.2**: Suppose we have two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. If P_1, P_2 are both above the x-axis ($y_1, y_2 \geq 0$) or both below the x-axis ($y_1, y_2 \leq 0$), then the "line" connecting them is a normal line and it's unique from normal Euclidean considerations. If $x_1 = x_2$, then the "line" is vertical and once again it is normal. If $y_1 = y_2$, then the "line" is horizontal and hence normal. Therefore, we may assume that one of the points is above the x-axis and the other below and that their x-coordinates don't match. Without loss of generality, we may assume that P_2 is the one above the x-axis. Therefore, we know that $y_2 > 0, y_1 < 0$. Since lines with negative slope are also kept, we can assume that $x_2 > x_1$ (otherwise, we've have a normal line connecting the points).

   Let's find a Moulton "line" that connects P_1 and P_2. Let $P = (x, 0)$ be the intersection with the x-axis and suppose the "line" starts with slope k and ends with slope $k/2$. Then we have

$$\begin{aligned}\frac{-y_1}{x-x_1} &= k\\ \frac{-y_2}{x-x_2} &= \frac{k}{2}.\end{aligned}$$

   Therefore

$$\frac{-y_1}{x-x_1} = k = 2\left(\frac{k}{2}\right) = 2\left(\frac{-y_2}{x-x_2}\right).$$

Hence

$$\begin{aligned} -y_1x + y_1x_2 &= -2y_2x + 2x_1y_2 \\ x(2y_2 - y_1) &= 2x_1y_2 - y_1x_2 \\ x &= \frac{2x_1y_2 - y_1x_2}{2y_2 - y_1}. \end{aligned}$$

Since $y_2 > 0$ and $y_1 < 0$, then $2y_2 - y_1 = y_2 + y_2 + (-y_1) > 0 + 0 + 0 = 0$. Therefore, the value of x exists and it's unique (since it only depends on x_1, y_1, x_2, y_2). One last thing to check is that $k > 0$. Note that

$$x - x_1 = \frac{2x_1y_2 - y_1x_2}{2y_2 - y_1} - x_1 = \frac{2x_1y_2 - y_1x_2 - 2x_1y_2 + x_1y_1}{2y_2 - y_1} = \frac{y_1(x_1 - x_2)}{2y_2 - y_1}.$$

Since $x_2 > x_1$, $2y_2 - y_1 > 0$, and $y_1 < 0$, then $x - x_1 > 0$. But $k = -y_1/(x - x_1)$. Since $x - x_1 > 0$ and $y_1 < 0$, then $-y_1/(x - x_1) > 0$. Therefore $k > 0$.

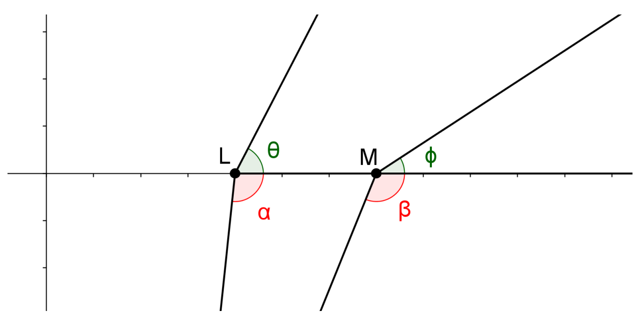
For **6.1.3**: Let ℓ_1, ℓ_2 be Moulton lines. Suppose that they intersect at P and at Q. From **6.1.2**, we know that there is a unique line containing P and Q. But ℓ_1 and ℓ_2 contain P, Q. Therefore $\ell_1 = \ell_2$. This shows that two lines cannot meet in two (or more) points.

We still need to show that two lines have to intersect. Let ℓ_1 and ℓ_2 be Moulton lines that are not parallel to each other. If both lines don't have positive slope, then they are normal lines and it follows that they intersect each other. We may assume that one of them has positive slope. Let ℓ_1 have slope $k > 0$ below the x-axis and $k/2$ above the x-axis. Let L be the intersection of ℓ_1 with the x-axis. Let p_1 to be the Euclidean line with slope $k/2$ that goes through L and p_2 the Euclidean line with slope k that goes through L.

We have two cases: either ℓ_2 is a normal (Euclidean) line or it's not.

Case I: Suppose ℓ_2 is a Euclidean line. Since ℓ_1 is not parallel to ℓ_2, then p_1 and p_2 intersect ℓ_2 at points A and B, respectively. Suppose ℓ_1 and ℓ_2 don't intersect. That means that P is below the x-axis and Q is above the x-axis. But then the slope of the line going through PQ is positive and for ℓ_2 to be a Euclidean line, the slope must be negative. Contradiction! Therefore, the lines intersect.

Case II: Suppose ℓ_2 is not a normal Euclidean line. Let M be the intersection of ℓ_2 with the x-axis. If $L = M$, then we're done, so we may assume $L \neq M$. Let $m > 0$ be the slope of ℓ_2 below the x-axis and $m/2$ above the x-axis. Because of symmetry, we may assume without loss of generality that M is to the right of L. We have two subcases, $k > m$ and $k < m$ ($k = m$ is the parallel case).

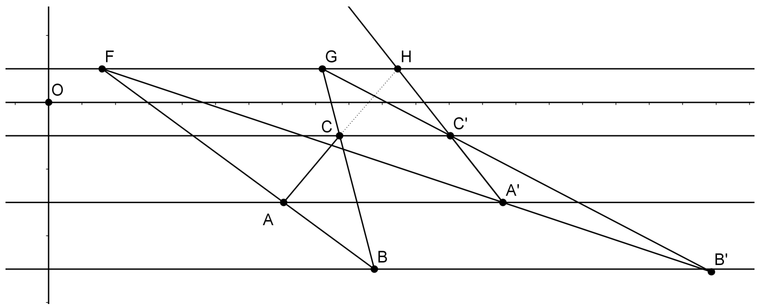


The angle at L between the x-axis and the ray with slope $k/2$ is $\theta = \arctan(k/2)$. The angle at M between the x-axis and the ray with slope $m/2$ is $\phi = \arctan(m/2)$. Note that since $k, m > 0$, we have $0 < \theta, \phi < 90°$. If $k < m$, then $k/2 < m/2$ and hence $\phi > \theta$, i.e., $\theta - \phi < 0$. But then $\theta + (180° - \phi) < 180°$ and by the Fifth postulate, that means that the lines intersect in that direction, i.e., they intersect above the x-axis. In that case, the Moulton lines intersect. If $k > m$, the relevant angles are below the x-axis. Let α be the angle at L between the ray below the x-axis of ℓ_1 and the x-axis. Let β be the angle at M between the ray below the x-axis of ℓ_2 and the x-axis. We have that $\alpha = 180° - \arctan(k)$ and $\beta = 180° - \arctan(m)$. The relevant angles are α

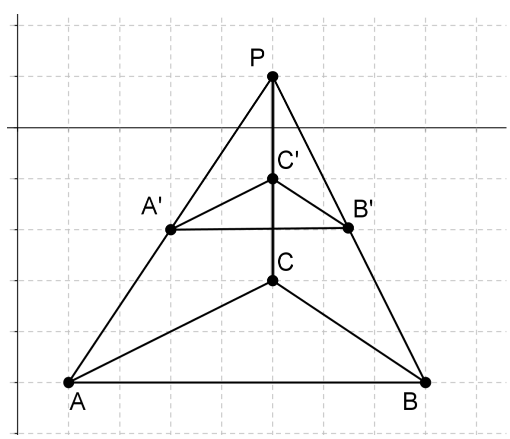
and $180° - \beta$. Since $k > m$, then $\alpha < \beta$. Then $\alpha + 180° - \beta < 180°$, so by the Fifth postulate, the lines intersect below the x-axis. Therefore ℓ_1 and ℓ_2 intersect.

3. Exercises 6.1.5 and 6.3.1.

Solution 3.



For **6.1.5**: The figure above is similar to Figure 6.5 of the textbook, with the difference that I labelled some points and I didn't draw the Moulton line from A to H. Line $\mathscr{L}$ in Figure 6.5 of the textbook would be the highest horizontal line in this figure, i.e., the line connecting F and G. The perspective triangles $\triangle ABC$, $\triangle A'B'C'$ in the figure have two pairs of corresponding sides that intersect at $\mathscr{L}$, namely F, G. These intersections are valid in the Euclidean plane and in the Moulton plane. Let H be the intersection of $A'C'$ with $\mathscr{L}$. By Desargues on the Euclidean plane, AC goes through H. If Desargues were to hold in the Moulton plane, then the line connecting A and C would go through H, or equivalently, the line connecting A to H must go through C. Let X be the point where the Moulton line from A to H crosses the x-axis. Then the slope m of AX is double the slope of XH. If we assume the line goes through C, then the slope of AX is the same as the slope of AC (because A and C are below the x-axis). But the Euclidean line from A to H goes through C and hence has slope m as well. This would imply that the slope from X to H is also m. But the Moulton line forces that slope to be $m/2$. Contradiction.

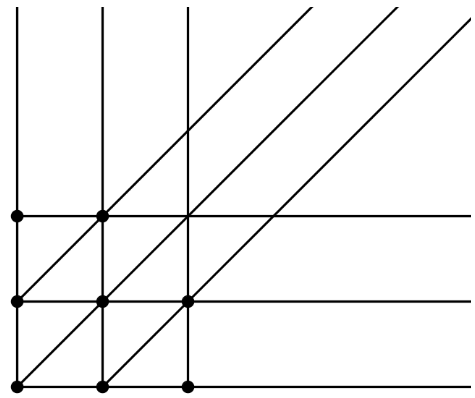


For **6.3.1**: The figure above is similar to Figure 6.16, except I drew it in the Euclidean plane. First note that $AB \| A'B'$, $AC \| A'C'$, and $BC \| B'C'$. Since all of the points are below the x-axis, then the parallel lines hold in the Moulton plane as well. This means that the intersection of the corresponding sides of the triangle meet at a line (in this case, the line at infinity). Therefore, the conditions for the converse of Desargues are satisfied. If we consider them in the Euclidean plane, then AA', BB', CC' intersect at a point P. Now note how BB' has negative slope and CC' is vertical. Therefore, those lines are the same in the Moulton plane as in the Eucldiean plane. So if $\triangle ABC$ and $\triangle A'B'C'$ are in perspective in the Moulton plane, the point of perspective must be P. Note that P is above the x-axis, therefore the Moulton line connecting A to P would break in two. For the same reasons as **6.1.5**, this line cannot go through A'. Therefore, the converse of Desargues is not true in the Moulton plane.

4. Exercises 6.2.1 and 6.2.2.

Solution 4.
For **6.2.1**: "If two triangles are in perspective from a point P (possibly at ∞), and if two pairs of corresponding sides are parallel, then the third pair of corresponding sides are also parallel."

For **6.2.2**: Figure 6.7 would look like



The little Desargues theorem implies that the diagonals in the figure are parallel to each other.

Figure 6.8 would look like:

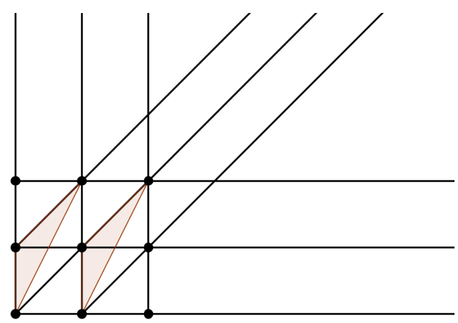
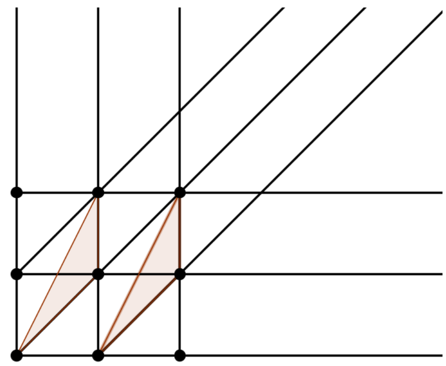


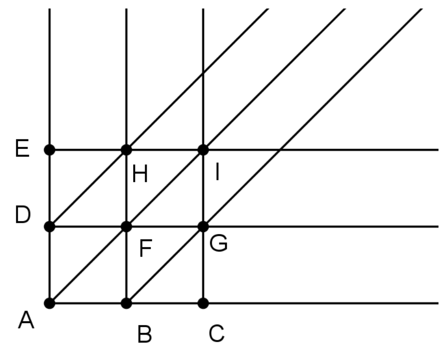
Figure 6.9 would look like:

5. Exercises 6.2.3 and 6.2.4.

Solution 5.
For **6.2.3**: In the drawing, the way the construction works is that we have A, B, D, F. Then we draw AF. Because $AD \| BF$ and $AB \| DF$, then $\mathscr{L}$ is the line at infinity. Therefore AF intersects $\mathscr{L}$ at ∞. We now draw the line from ∞ to D. This means we draw a line parallel to AF through D. This line intersects BF at H. Then we draw a line from $AB \cap DF = \infty$ to H. Therefore, we draw a line parallel to AB through H. The intersection with AD gives us E. Then we want to draw the line from $AD \cap BF = \infty$ to $I = EH \cap AF$. Therefore, we want to draw a line parallel to AD through I. This line intersects DF at G and AB at C. The coincidence is that $BG \| AF$.

For **6.2.4**: By little Desargues using the "new" Figure 6.8 with the labels from here, we have that since $DH \| FI$, $AD \| BF$ and $\triangle ADH$ and $\triangle BFI$ are in perspective (because $AB \| DF \| HI$), then $AH \| BI$.

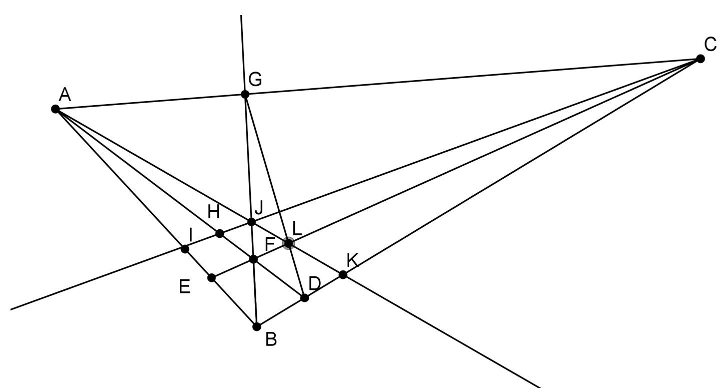




Using the "new" Figure 6.9 with the labels from this problem, we have $\triangle AHF$ and $\triangle BIG$ are in perspective because $AB \| FG \| HI$. Using little Desargues, since $AH \| BI$ and $HF \| IG$, then $AF \| BG$. But that's what we wanted to prove.

6. Exercise 6.2.5.

   Solution 6. Consider the figure:



   Place the x-axis above E and below G. Since BF and DL have negative slope, their intersection is G in the Euclidean plane and in the Moulton plane. However, since E is below the x-axis and G is above the x-axis, the Moulton line connecting them must split in two. Therefore, it can't go through H (since the Euclidean line EH goes through G).

7. Exercises 6.3.2 and 6.3.3.

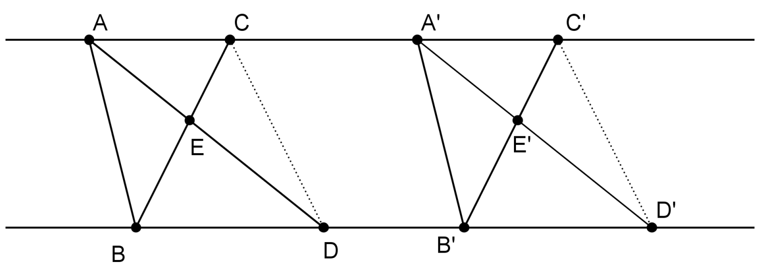
Solution 7.

For **6.3.2**: "Suppose triangles $\triangle ABC$ and $\triangle A'B'C'$ satisfy that $AB \cap A'B'$, $BC \cap B'C'$ and $A'C' \cap AC$ are in a line $\mathscr{L}$ that goes through a point P. Suppose also that $AA' \cap CC' = P$. Then the triangles are in perspective with respect to P."

Proof: One easy proof is to simply invoke the converse of Desargues. Since the three corresponding pairs intersect at points at a line $\mathscr{L}$, then the triangles are in perspective. Since $AA' \cap CC' = P$, then they could only be in perspective with respect to P.

The exercise asks for a proof using little Desargues, so let's come up with one. Let $D = BC \cap B'C'$. Let B'' be the intersection of BP with DC'. By construction, $\triangle ABC$ and $\triangle A'B''C'$ are in perspective. We also know that $B''C' \cap BC = D$ is in $\mathscr{L}$ and that $AC \cap A'C'$ is also in $\mathscr{L}$. Therefore, by the Little Desargues Theorem, $AB \cap A'B''$ is also in $\mathscr{L}$. Let $E = \mathscr{L} \cap AB$. Then $A'B'$ goes through E by our initial assumption (that the corresponding sides intersect in $\mathscr{L}$). But $A'B''$ also goes through E. Therefore B' and B'' are both in the line EA' and in the line DC'. Therefore $B' = B'' = EA' \cap DC'$. That means $\triangle ABC$ and $\triangle A'B'C'$ are in perspective with respect to point P.

For **6.3.3**: Suppose A, C, A', C' is in a line ℓ_1 and B, D, B', D' is in a line ℓ_2. Furthermore, suppose that $\ell_1 \| \ell_2$, $AB \| A'B'$, $AD \| A'D'$, $BC \| B'C'$. We want to show that $CD \| C'D'$.

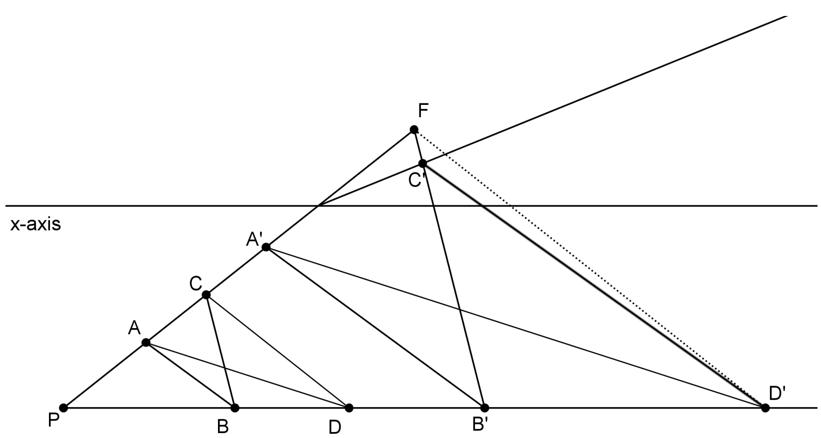


Let E be the intersection of AD and BC. Let E' be the intersection of $A'D'$ and $B'C'$. $AE \| A'E'$, $AB \| A'B'$, $BE \| B'E'$, so they all meet at the line at infinity. Since $AA' \| BB'$, then those lines also meet at the line at infinity. Then, by the converse of little Desargues, the triangles $\triangle ABE$ and $\triangle A'B'E'$ are in perspective with respect to the point at infinity. Therefore $EE' \| AA' \| BB'$.

Now, since $CC' \| AA' \| EE' \| BB' \| DD'$, then $\triangle CED$ and $\triangle C'E'D'$ are in perspective. Since $EC \| E'C'$ and $ED \| E'D'$, then the remaining corresponding sides must be parallel too. Therefore $CD \| C'D'$.

8. Exercise 6.3.4.

Solution 8.



In the diagram, the quadrilaterals are in perspective with respect to P. We also have $AB \| A'B'$, $BC \| B'C'$, $AD \| A'D'$. The question is whether this implies that $CD \| C'D'$. Note that CD and $C'D'$ have negative slope, so they are normal Euclidean lines.

Let F be the intersection of $B'C'$ with the Euclidean line PA. From the Scissors Theorem, we know that $FD' \| CD$ (because $AB \| A'B', BC \| B'C' \| B'F, AD \| A'D'$, and $ABCD$ and $A'B'FD'$ are quadrilaterals in perspective). Since $D'F \| CD$ and $F \neq C'$, then $D'C'$ is not parallel to CD (otherwise, D', F, C' would be collinear, but that means that $B' = D'$ which is clearly not true in that diagram).