Geometry
Homework 8 Solutions

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1. Exercises 6.6.1, 6.6.2, and 6.6.3.

   Solution 1.

   For **6.6.1**:

$$\begin{aligned}\left|\begin{pmatrix} a+bi & c+id \\ -c+id & a-ib \end{pmatrix}\right| &= (a+ib)(a-ib)-(c+id)(-c+id)\\ &= a^2-(i^2)b^2-(i^2d^2-c^2)\\ &= a^2-(-b^2)+c^2-(-d^2)\\ &= a^2+b^2+c^2+d^2.\end{aligned}$$

   Suppose $\mathbf{q}$ is not zero. Then at least one of a,b,c,d is not zero. Therefore $a^2+b^2+c^2+d^2 \neq 0$. Any matrix with nonzero determinant has an inverse, therefore $\mathbf{q}^{-1}$ exists.

   For **6.6.2**: Let $\mathbf{s} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{t} = \begin{pmatrix} i & 2 \\ -2 & -i \end{pmatrix}$. Note that $\mathbf{s}$ is of the same form as $\mathbf{q}$ where $a=1, b=0, c=2, d=0$, and $\mathbf{t}$ would have $a=0, b=1, c=2, d=0$. Then

$$\mathbf{st} = \begin{pmatrix} -4+i & 2-2i \\ -2-2i & -4-i \end{pmatrix},$$

   and

$$\mathbf{ts} = \begin{pmatrix} -4+i & 2+2i \\ -2+2i & -4-i \end{pmatrix}.$$

   They are different because $2-2i \neq 2+2i$.

   For **6.6.3**:

$$\mathbf{i}^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} i^2 & 0 \\ 0 & (-i)^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{1}.$$

$$\mathbf{j}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{1}.$$

$$\mathbf{k}^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i^2 & 0 \\ 0 & i^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{1}.$$

$$\mathbf{ijk} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \mathbf{k}^2 = -\mathbf{1}.$$

2. Exercise 6.7.1.

   Solution 2. We are assuming that $a(b+c) = ab+ac$. We also know that the operation is commutative, therefore $a(b+c) = (b+c)a$, $ab = ba$, and $ac = ca$. Therefore

$$(b+c)a = a(b+c) = ab+ac = ba+ca.$$

3. Exercise 6.7.2.

Solution 3. We'll use that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$, that $\mathbf{ij} = \mathbf{k}$, $\mathbf{ji} = -\mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, $\mathbf{kj} = -\mathbf{i}$, $\mathbf{ki} = \mathbf{j}$, and $\mathbf{ik} = -\mathbf{j}$. Suppose we have the quaternions $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Then, there exist real numbers $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4$ such that

$$\begin{aligned}
\mathbf{a} &= a_1\mathbf{1} + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k} \\
\mathbf{b} &= b_1\mathbf{1} + b_2\mathbf{i} + b_3\mathbf{j} + b_4\mathbf{k} \\
\mathbf{c} &= c_1\mathbf{1} + c_2\mathbf{i} + c_3\mathbf{j} + c_4\mathbf{k}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{ab} &= (a_1\mathbf{1} + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k})(b_1\mathbf{1} + b_2\mathbf{i} + b_3\mathbf{j} + b_4\mathbf{k}) \\
&= a_1b_1 + a_1b_2\mathbf{i} + a_1b_3\mathbf{j} + a_1b_4\mathbf{k} + \\
&\qquad + a_2b_1\mathbf{i} + a_2b_2\mathbf{i}^2 + a_2b_3\mathbf{ij} + a_2b_4\mathbf{ik} \\
&\qquad + a_3b_1\mathbf{j} + a_3b_2\mathbf{ji} + a_3b_3\mathbf{j}^2 + a_3b_4\mathbf{jk} \\
&\qquad + a_4b_1\mathbf{k} + a_4b_2\mathbf{ki} + a_4b_3\mathbf{kj} + a_4b_4\mathbf{k}^2 \\
&= (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4)\mathbf{1} + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)\mathbf{i} \\
&\qquad + (a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2)\mathbf{j} + (a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)\mathbf{k}.
\end{aligned}$$

Analogously

$$\begin{aligned}
\mathbf{ac} &= (a_1\mathbf{1} + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k})(c_1\mathbf{1} + c_2\mathbf{i} + c_3\mathbf{j} + c_4\mathbf{k}) \\
&= (a_1c_1 - a_2c_2 - a_3c_3 - a_4c_4)\mathbf{1} + (a_1c_2 + a_2c_1 + a_3c_4 - a_4c_3)\mathbf{i} \\
&\qquad + (a_1c_3 - a_2c_4 + a_3c_1 + a_4c_2)\mathbf{j} + (a_1c_4 + a_2c_3 - a_3c_2 + a_4c_1)\mathbf{k}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{ab} + \mathbf{ac} &= (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 + a_1c_1 - a_2c_2 - a_3c_3 - a_4c_4)\mathbf{1} \\
&\quad + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3 + a_1c_2 + a_2c_1 + a_3c_4 - a_4c_3)\mathbf{i} \\
&\quad + (a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2 + a_1c_3 - a_2c_4 + a_3c_1 + a_4c_2)\mathbf{j} \\
&\quad + (a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1 + a_1c_4 + a_2c_3 - a_3c_2 + a_4c_1)\mathbf{k} \\
&= (a_1(b_1 + c_1) - a_2(b_2 + c_2) - a_3(b_3 + c_3) - a_4(b_4 + c_4)\mathbf{1} \\
&\quad + (a_1(b_2 + c_2) + a_2(b_1 + c_1) + a_3(b_4 + c_4) - a_4(b_3 + c_3)\mathbf{i} \\
&\quad + (a_1(b_3 + c_3) - a_2(b_4 + c_4) + a_3(b_1 + c_1) + a_4(b_2 + c_2)\mathbf{j} \\
&\quad + (a_1(b_4 + c_4) + a_2(b_3 + c_3) - a_3(b_2 + c_2) + a_4(b_1 + c_1)\mathbf{k} \\
&= (a_1\mathbf{1} + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k})((b_1 + c_1)\mathbf{1} + (b_2 + c_2)\mathbf{i} + (b_3 + c_3)\mathbf{j} + (b_4 + c_4))\mathbf{k} \\
&= \mathbf{a}(\mathbf{b} + \mathbf{c}).
\end{aligned}$$

4. Exercises 7.6.3, 7.6.4, and 7.6.5.

Solution 4.

For **7.6.3**: Since $\mathbf{1}$ and $\mathbf{i}$ are perpendicular to each other and of length 1, then by the Pythagorean Theorem

$$|\mathbf{1} + \mathbf{i}| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Analogously for $|\mathbf{1} - \mathbf{i}| = \sqrt{2}$. Then

$$|\mathbf{1} - \mathbf{i}^2| = |(\mathbf{1} - \mathbf{i})(\mathbf{1} + \mathbf{i})| = |\mathbf{1} - \mathbf{i}| \cdot |\mathbf{1} + \mathbf{i}| = \sqrt{2} \cdot \sqrt{2} = 2.$$

For **7.6.4**: Since $|\mathbf{i}| = 1$, then $|\mathbf{i}^2| = |\mathbf{i}| \cdot |\mathbf{i}| = 1 \cdot 1 = 1$. Therefore $\mathbf{1} - \mathbf{i}^2$ is at distance 1 from $\mathbf{1}$.

By the triangle inequality we have

$$2 = |\mathbf{1} - \mathbf{i}^2| \leq \mathbf{1} + |\mathbf{i}^2| = 1 + 1 = 2.$$

The only way to have equality is if the vectors are pointing in the same direction. Therefore $\mathbf{1} - \mathbf{i}^2 = c\mathbf{1}$. Since the length of the vector is 2, then $c = 2$. Therefore

$$\begin{aligned} \mathbf{1} - \mathbf{i}^2 &= 2\mathbf{1} \\ \mathbf{i}^2 &= -\mathbf{1}. \end{aligned}$$

For **7.6.5**: Exercises 7.6.3 and 7.6.4 only used that $\mathbf{i}$ is perpendicular to $\mathbf{1}$ and has length 1. Therefore, if $\mathbf{u}$ is any vector of length 1 perpendicular to $\mathbf{1}$, then $\mathbf{u}^2 = -\mathbf{1}$.

5. Exercises 7.6.6 and 7.6.7.

Solution 5.

For **7.6.6**: Since $\mathbf{i} \perp \mathbf{j}$, then $\mathbf{i}^2 \perp \mathbf{ij}$, i.e, $-\mathbf{1} \perp \mathbf{ij}$. But $\mathbf{1}$ and $-\mathbf{1}$ are parallel vectors, so $\mathbf{1} \perp \mathbf{ij}$.

For **7.6.7**: But then $(\mathbf{ij})^2 = -\mathbf{1}$. Throughout these exercises, we've been assuming that there is a commutative product for the vectors in $\mathbb{R}^n$ (for some $n \geq 3$). Therefore $\mathbf{ij} = \mathbf{ji}$. We also know that $\mathbf{j}^2 = -\mathbf{1}$ and $\mathbf{i}^2 = -\mathbf{1}$. Therefore

$$-\mathbf{1} = (\mathbf{ij})^2 = (\mathbf{ij})(\mathbf{ji}) = \mathbf{i}(\mathbf{j})^2\mathbf{i} = \mathbf{i}(-\mathbf{1})\mathbf{i} = -\mathbf{1}\mathbf{i}^2 = \mathbf{1}.$$

Contradiction! Therefore we can't satisfy all of the field axioms for an operation in $\mathbb{R}^n$, for $n \geq 3$.

6. Exercises 8.1.3 and 8.1.4.

Solution 6.

For **8.1.3**: First do $z \to z + 1$, then do $z \to 2z$ (or do them in the other order).

For **8.1.4**: You can shift horizontally from one semicircle to the other by translating the distance between the centers. So if you have your first circle with center at $(x_1, 0)$ and the second one at $(x_2, 0)$, then you let $\ell = x_2 - x_1$ and you do the transformation $z \to z + \ell$. After that, you need only scale. If the first semicircle has radius r_1 and the second one r_2, then you create the transformation $z \to \frac{r_2}{r_1} z$. Note that this transformation is legal because $r_2/r_1 > 0$.

7. In the statement of Pascals Theorem (problem 6 in Midterm 2) all six points are distinct. However, when two points are the same on a circle, we can still think of them as distinct but "infinitesimally" close. In this way the line they determine is the tangent to the conic at their common position.

   (a) State the analogue of Pascals Theorem in the case when just two of the points of the hexagon, say A and F, coincide on the circle. Draw a picture.

   (b) State the analogue of Pascals Theorem when $E = F$ and $C = D$. Draw a picture.

Solution 7.

(a) Given five points A, B, C, D, E on a circle. Let $G = AB \cap ED$, $H = BC \cap AE$, and $K = CD \cap$ tangent at A. Then G, H, K are collinear. Figure 1 shows the configuration.

(b) Given four points A, B, C, E on a circle. Let $G = AB \cap CE$, H be the intersection of BC with the tangent at E, and K be the intersection of AE with the tangent at C. Then G, H, K are collinear. Figure 2 shows the configuration.

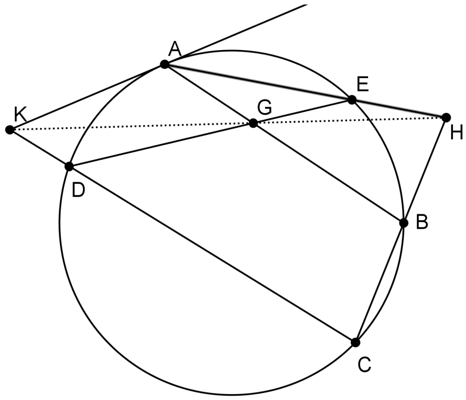


Figure 1: Pascal's Hexagon when $A = F$.

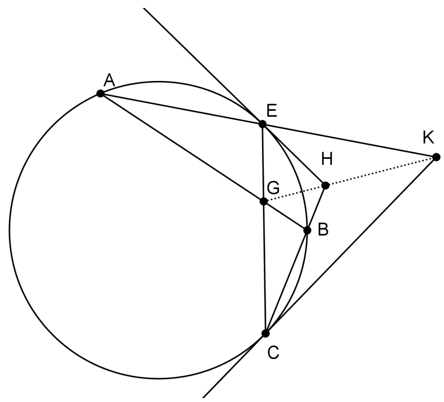
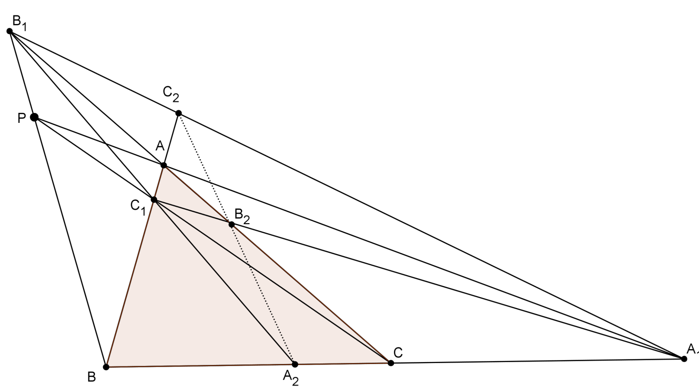


Figure 2: Pascal's Hexagon with $C = D$ and $E = F$.

8. Lines AP, BP and CP meet the sides of triangle $\triangle ABC$ at points A_1, B_1 and C_1, respectively. Suppose that lines B_1C_1, C_1A_1, A_1B_1 intersect BC, CA, AB at points A_2, B_2, C_2, respectively. Prove that the points A_2, B_2 and C_2 lie on a line.



Solution 8. $\triangle ABC$ and $\triangle A_1B_1C_1$ are in perspective with respect to P. Therefore, by Desargues, the intersections of the corresponding sides are collinear. We have that $A_2 = BC \cap B_1C_1$, $B_2 = AC \cap A_1C_1$, and $C_2 = AB \cap A_1B_1$, i.e., they are the intersections of the corresponding sided. Therefore A_2, B_2, C_2 are collinear.