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The Box Principle

The simplest version of Dirichlet's box principle reads as follows:

If $(n + 1)$ pearls are put into n boxes, then at least one box has more than one pearl.

This simple combinatorial principle was first used explicitly by Dirichlet (1805–1859) in number theory. In spite of its simplicity it has a huge number of quite unexpected applications. It can be used to prove deep theorems. F.P. Ramsey made vast generalizations of this principle. The topic of *Ramsey Numbers* belongs to the deepest problems of combinatorics. In spite of huge efforts, progress in this area is very slow.

It is easy to recognize if the box principle is to be used. Every existence problem about finite and, sometimes, infinite sets is usually solved by the box principle. The principle is a pure existence assertion. It gives no help in finding a multiply occupied box. The main difficulty is the identification of the *pearls* and the *boxes*.

For a warmup, we begin with a dozen simple problems without solutions:

1. Among three persons, there are two of the same sex.

2. Among 13 persons, there are two born in the same month.

3. Nobody has more than 300,000 hairs on his head. The capital of Sikinia has 300,001 inhabitants. Can you assert with certainty that there are two persons with the same number of hairs on their heads?

4. How many persons do you need to be sure that 2 (3, q) persons have the same birthday?

5. If $qs + 1$ pearls are put into s boxes, then at least one box has more than q pearls.
6. A line l in the plane of the triangle ABC passes through no vertex. Prove that it cannot cut all sides of the triangle.
7. A plane does not pass through a vertex of a tetrahedron. How many edges can it intersect?
8. A target has the form of an equilateral triangle with side 2.
   (a) If it is hit 5 times, then there will be two holes with distance ≤ 1.
   (b) It is hit 17 times. What is the minimal distance of two holes at most?
9. The decimal representation of a/b with coprime a, b has at most period $(b-1)$.
10. From 11 infinite decimals, we can select two numbers a, b so that their decimal representations have the same digits at infinitely many corresponding places.
11. Of 12 distinct two-digit numbers, we can select two with a two-digit difference of the form aa.
12. If none of the numbers $a, a+d, \ldots, a+(n-1)d$ is divisible by n, then d and n are coprime.

The next eleven examples show typical applications of the box principle.

E1. *There are n persons present in a room. Prove that among them there are two persons who have the same number of acquaintances in the room.*

Solution. A person (pearl) goes into box #i if she has i acquaintances. We have n persons and n boxes numbered $0, 1, \ldots, n-1$. But the boxes with the numbers 0 and $n-1$ cannot both be occupied. Thus, there is at least one box with more then one pearl.

E2. *A chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but not more then 132 games. Prove that there is a sequence of successive days on which he plays exactly 21 games.*

Solution. Let a_i be the number of games played until the ith day inclusive. Then

$$1 \le a_1 < \ldots < a_{77} \le 132 \Rightarrow 22 \le a_1 + 21 < a_2 + 21 < \ldots < a_{77} + 21 \le 153$$

Among the 154 numbers $a_1, \ldots, a_{77}, a_1 + 21, \ldots, a_{77} + 21$ there are two equal numbers. Hence there are indices i, j, so that $a_i = a_j + 21$. The chessmaster has played exactly 21 games on the days $\#j+1, j+2, \ldots, i$.

E3. *Let $a_1, a_2, \ldots, a_n$ be n not necessarily distinct integers. Then there always exists a subset of these numbers with sum divisible by n.*

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Solution. We consider the n integers

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \ldots, \quad s_n = a_1 + a_2 + \cdots + a_n.$$

If any of these integers is divisible by n, then we are done. Otherwise, all their remainders are different modulo n. Since there are only $n-1$ such remainders, two of the sums, say s_p and s_q with $p < q$, are equal modulo n, that is, the following difference is divisible by n.

$$s_q - s_p = a_{p+1} + \ldots + a_q.$$

This proof contains an important motive with many applications in number theory, group theory, and other areas.

E4. *One of $(n+1)$ numbers from $\{1, 2, \ldots, 2n\}$ is divisible by another.*

Solution. We select $(n+1)$ numbers $a_1, \ldots, a_{n+1}$ and write them in the form $a_i = 2^k b_i$ with b_i odd. Then we have $(n+1)$ odd numbers $b_1, \ldots, b_{n+1}$ from the interval $[1, 2n-1]$. But there are only n odd numbers in this interval. Thus two of them p, q are such that $b_p = b_q$. Then one of the numbers a_p, a_q is divisible by the other.

E5. *Let $a, b \in \mathbb{N}$ be coprime. Then $ax - by = 1$ for some $x, y \in \mathbb{N}$.*

Solution. Consider the remainders mod b of the sequence $a, \ldots, (b-1)a$. The remainder 0 does not occur. If the remainder 1 would not occur either, then we would have positive integers $p, q, 0 < p < q < b$, so that $pa \equiv qa \pmod{b}$. But a and b are coprime. Hence we have $b|q-p$. This is a contradiction since $0 < q - p < b$. Thus there exists an x so that $ax \equiv 1 \pmod{b}$, that is, ax=1+by, or $ax - by = 1$.

E6. Erdős and Szekeres. *The positive integers 1 to 101 are written down in any order. Prove that you can strike 90 of these numbers, so that a monotonically increasing or decreasing sequence remains.*

Solution. We prove a generalization: For $n \geq (p-1)(q-1)+1$ every sequence of n integers contains either a monotonically increasing subsequence of length p or a monotonically decreasing subsequence of length q.

We assign the maximal length L_m of a monotonically increasing sequence with last element m and the maximal length R_m of a monotonically decreasing sequence beginning with m to any number m in the sequence.

This assignment has the property that, for two different numbers m and k there must be $L_m \neq L_k$ or $R_m \neq R_k$. This follows easily from the fact that either $m > k$ or $m < k$. All pairs (L_m, R_m) with $m = 1, 2, \ldots, n$ are distinct. Assuming that no such subsequences exist, L_m can assume only the values $1, 2, \ldots, p-1$ and R_m only the values $1, 2, \ldots, q-1$. This gives $(p-1)(q-1)$ different boxes for the pairs. But $n \geq (p-1)(q-1)+1$ and the box principle leads to a contradiction.

E7. *Five lattice points are chosen in the plane lattice. Prove that you can always choose two of these points such that the segment joining these points passes through*