Cardinality Lectures

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1 Definition of cardinality

The cardinality of a set is a measure of the size of a set. When a set A is finite, its cardinality is the number of elements of the set, usually denoted by $|A|$. When the set is infinite, comparing if two sets have the "same size" is a little different. Georg Cantor in the late 1800s came out with the following idea to compare infinite sets:

Definition 1. *Let A and B be sets. We say $|A| = |B|$ if and only if there exists a bijective function $f : A \to B$.*

Note that the definition also works for finite sets, because if A and B are finite sets and $f : A \to B$ is a bijection, then $|A| = |B|$.

Here are two other definitions that work for finite and infinite sets:

Definition 2. *Let A and B be sets. We say $|A| \geq |B|$ if and only if there exists an onto function $f : A \to B$ (or B is empty).*

Definition 3. *Let A and B be sets. We say $|A| \leq |B|$ if there exists a one-to-one function $f : A \to B$ (or if A is empty).*

These definitions extend naturally to the following:

- $|A| > |B|$ if $|A| \geq |B|$ and $|A| \neq |B|$, i.e., there exists an onto function from A to B, but no bijection exists from A to B.
- $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$, i.e., there exists a one-to-one function from A to B, but no bijection exists from A to B.

Note: For simplicity, in the rest of the article we will avoid considering the empty set. Adding an extra line to prove things for the empty set is easy in all the coming propositions and theorems, but to be brief I will refrain from adding those lines in the proofs.

2 Infinite sets versus finite sets

Let's check if our definitions pass the "eyeball-test" by checking if an infinite set is indeed "larger" than a finite set:

Proposition 1. *Let A be an infinite set and B be a finite set. Then $|A| > |B|$.*

Proof. To prove the proposition we need to show that an onto function exists from A to B, yet there is no bijection from A to B. Let's start by building an onto function. Since B is finite, $|B| = n$ for some $n \in \mathbb{N}$, therefore $B = \{b_1, b_2, \ldots, b_n\}$ for some $b_1, b_2, \ldots b_n$. Since A is infinite, it has at least n elements, label these n elements $a_1, a_2, \ldots a_n$.

Let $f : A \to B$ be defined as

$$f(a) = \begin{cases} b_i & \text{if } a = a_i \text{ for } i = 1, 2, \ldots, n \\ \\ b_1 & \text{otherwise.} \end{cases}$$

Clearly f is a function and its image is B, so f is onto. This proves that $|A| \geq |B|$.

Now, let's prove that $|A| \neq |B|$. For the sake of contradiction, suppose $|A| = |B|$, i.e., suppose that there exists a bijection $g : A \to B$. Since A is infinite, then it has at least $n + 1$ elements, label these $a_1, a_2, \ldots, a_n, a_{n+1}$. Since g is one-to-one, then $g(a_1), g(a_2), \ldots g(a_{n+1})$ are all different, so the size of B must be at least $n + 1$. But the size of B is n. This contradicts our assumption that $|A| = |B|$.

□

Since cardinality tries to measure size, it would be nice to show that a subset of another set has smaller cardinality. That's what the next proposition says:

Proposition 2. *If A and B are sets and $A \subseteq B$, then $|A| \leq |B|$.*

Proof. Let $f : A \to B$ be the function $f(a) = a$ for $a \in A$. f is one-to-one because $f(a) = f(b) \implies a = b$. Since f is one-to-one by definition $|A| \leq |B|$.

□

3 Examples of bijections between infinite sets

Proposition 3. *The interval* $(0, 1)$ *has the same cardinality as the interval* $(0, 7)$.

Proof. Let $f : (0, 1) \to (0, 7)$ be defined by $f(x) = 7x$. Let's prove f is a bijection. To prove that f is a bijection we must show

(a) that f is one-to-one, and

(b) that f is onto.

Let's show that f is one-to-one: Let $f(a) = f(b)$, then $7a = 7b$, so $a = b$. Therefore f is one-to-one.

Let's show that f is onto: Suppose $b \in (0, 7)$. Let $a = b/7$, then $f(a) = 7(b/7) = b$ and $a \in (0, 1)$, so b is in the image of f. Since b is arbitrary, then f is onto.

Since f is one-to-one and f is onto, f is a bijection. Since f is a bijection, the intervals $(0, 1)$ and $(0, 7)$ have the same cardinality.

□

Proposition 4. *The interval* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *has the same cardinality as* $\mathbb{R}$.

Proof. To prove that the sets have the same cardinality we must find a bijection between them. I claim that $f(x) = tanx$ is a bijection from $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to $\mathbb{R}$. Let's start by showing it is one-to-one. Suppose $f(a) = f(b)$. Then $\tan a = \tan b$ where a and b are in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore a and b are angles between $-\pi/2$ and $\pi/2$. Since $\tan a = \tan b$ and they are angles with that restriction, then $a = b$.

Now let's show f is onto. Let x be a real number. We need to show there exists an $a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $f(a) = x$, i.e., such that $\tan a = x$. Let $a = \arctan x$. Then $\tan a = x$. We need only verify that a is between $-\pi/2$ and $\pi/2$. But the arctan function has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so a must be between $-\pi/2$ and $\pi/2$.

This concludes the proof.

A more illustrative proof is the graph of $f(x) = \tan x$ (Figure 1).

□

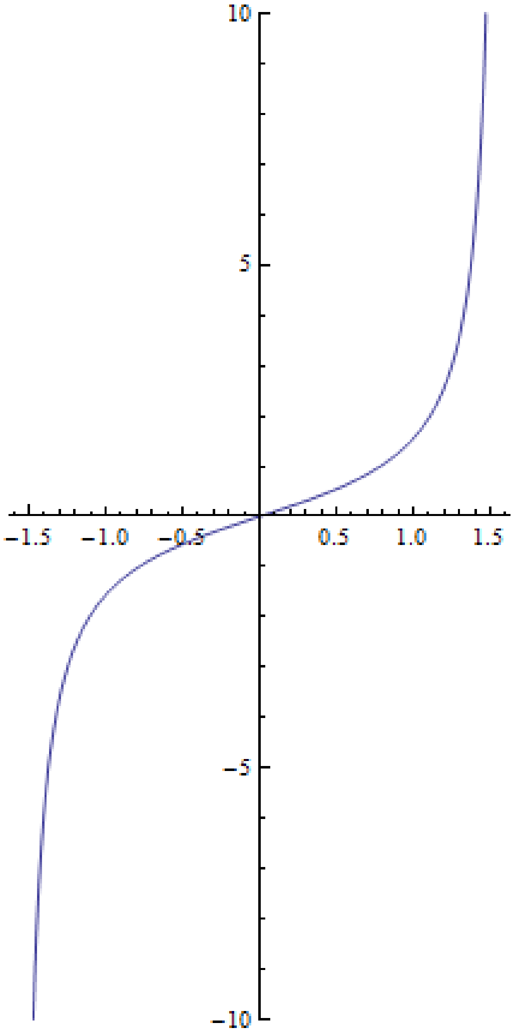

Figure 1: Note that in the graph, any horizontal line intersects the graph at most once, which means the function is one-to-one. Also note that every horizontal line intersects the graph, so every y value has at least one x value. This means the function is onto.

Proposition 5. *The set of even integers has the same cardinality as the set of integers.*

Proof. Let $2\mathbb{Z}$ be the set of even integers. Let $f : \mathbb{Z} \to 2\mathbb{Z}$ be the function $f(n) = 2n$.

f is one-to-one because if $f(m) = f(n)$, then $2m = 2n$ and hence $m = n$.

f is onto because if n is an even integer, then $n = 2m$ for some integer m (by the definition of even integer) and hence $f(m) = n$. So all even integers are in the image of f, so f is onto.

□

4 Cardinality is an equivalence relation

In this section we will prove that the cardinality relation is an equivalence relation. We say two sets A and B are related by cardinality if $|A| = |B|$.

First note that any set A satisfies $|A| = |A|$, because $f : A \to A$ defined by $f(a) = a$ is a bijection. Therefore the cardinality relation is reflexive.

That cardinality is symmetric is not obvious, but it's not hard to prove. Suppose $|A| = |B|$, therefore there exists a bijection $f : A \to B$. Since f is one-to-one then $f^{-1} : Im(f) \to A$ is a bijection (it's one-to-one and onto by definition of f^{-1}). Since f is onto, then $Im(f) = B$, therefore f^{-1} is a bijection from B to A. Therefore $|B| = |A|$.

Transitivity is a bit harder to prove. Proving transitivity boils down to showing that given a bijection from A to B and another bijection from B to C, there exists a bijection from A to C. We prove that in the following proposition (where we go even further and prove that the composition of two bijections is a bijection):

Proposition 6. *Let $f : A \to B$ and $g : B \to C$ be bijections, then $g \circ f : A \to C$ is a bijection.*

Proof. Let's start by proving that $g \circ f$ is one-to-one. Suppose $g \circ f(a) = g \circ f(b)$. Then $g(f(a)) = g(f(b))$. Since g is one-to-one, this implies $f(a) = f(b)$. Since f is one-to-one, $a = b$. Therefore $g \circ f$ is one-to-one.

Let's now prove that $g \circ f$ is onto. Suppose $c \in C$. Since $c \in C$ and g is onto, there exists $b \in B$ such that $g(b) = c$. Since $b \in B$ and f is onto, there exists an $a \in A$ such that $f(a) = b$. Therefore $g(f(a)) = c$, i.e., $g \circ f(a) = c$.

Since $g \circ f$ is one-to-one and onto, it is a bijection.

□

Since cardinality is reflexive, symmetric and transitive, then it is an equivalence relation.