Section 17 Solutions

$\boxed{17.3}$

a) $\binom{6}{3} = \frac{6\times5\times4}{3\times2} = 5\times4 = \boxed{20}$

b) $\binom{6}{3}(2^3)(-3)^3 = (20)(-6)^3 = \boxed{-4320}$

c) $\binom{20}{3} + \binom{20}{3}(-1)^{17} = \boxed{0}$

d) $\binom{6}{3} = \boxed{20}$

e) $\binom{7}{3} = \frac{7\times6\times5}{6} = \boxed{35}$

$\boxed{17.4}$ There are $\binom{30}{2} = \frac{30\cdot29}{2}$ ways of picking 2.

There are $2\binom{30}{2} = 30\cdot29$ ways of putting them on.

$\boxed{17.5}$ $\binom{20}{2} = \frac{20\times19}{2} = \boxed{190}$

$\boxed{17.8}$ a) $50!$

b) $\binom{50}{10}$ ← we don't care about the order.

c) $50\times49\times48$ ← we care about the order since gold, silver and bronze are delivered.

$\boxed{17.11}$ $\binom{n}{4}$

$\boxed{17.16}$ Proof: $k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-k)!}$

$= n\frac{(n-1)!}{(k-1)!(n-1-(k-1))!} = n\binom{n-1}{k-1}$ ☒

Proof 2: $k\binom{n}{k}$ counts the number of ways of selecting k numbers out of $\{1,2,3,\dots,n\}$ and then calling one of the numbers "king".

Since one out of $\{1,2,\dots,n\}$ is "king" there are n ways of choosing the king. We select $k-1$ out of the $n-1$ numbers left to be his citizens. So $n\binom{n-1}{k-1}$ is the same as $k\binom{n}{k}$ ∎

$\boxed{17.21}$ $$\binom{2n}{n} \approx \frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{2\pi(2n)}}{\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}}$$

$$= \boxed{\frac{2^{2n}}{\sqrt{\pi n}}}$$

Prove $\binom{2n}{n}$ is less than 4^n.

$$\binom{2n}{n} < \binom{2n}{0}+\binom{2n}{1}+\binom{2n}{2}+\dots+\binom{2n}{2n-1}+\binom{2n}{2n} = 2^{2n} = 4^n$$

so $\binom{2n}{n} < 4^n$ for $n \geq 1$.

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Homework 5
SOLUTIONS

(22.4) Prove by induction

a) Prove $1+4+\dots+(3n-2) = \frac{n(3n-1)}{2}$

Proof: For $n=1$ $\quad 1 = \frac{1(2)}{2}$ ✓

Suppose $1+4+\dots+(3k-2) = \frac{k(3k-1)}{2}$

Now $1+4+\dots+(3k-2)+3k+1 = \frac{k(3k-1)}{2} + (3k+1)$

$$= \frac{k(3k-1)+2(3k+1)}{2} = \frac{3k^2-k+6k+2}{2} = \frac{3k^2+5k+2}{2}$$

$$= \frac{(3k+2)(k+1)}{2} = \frac{(k+1)(3(k+1)-1)}{2}$$

Therefore $1+4+\dots+(3(k+1)-2) = \frac{(k+1)(3(k+1)-1)}{2}$

Therefore by induction $1+4+\dots+(3n-2) = \frac{n(3n-1)}{2}$ $\boxtimes$

b) Prove $1^3+2^3+\dots+n^3 = \frac{n^2(n+1)^2}{4}$.

Proof: By induction.

For $n=1$, $\quad 1 = \frac{1^2(1+1)^2}{4} = 1$ ✓

Assume for $n=k$, i.e. $1^3+2^3+\dots+k^3 = \frac{k^2(k+1)^2}{4}$

Now $1^3+2^3+3^3+\dots+k^3+(k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$

$$= \frac{(k+1)^2}{4}\left(k^2+4(k+1)\right) = \frac{(k+1)^2}{4}\left(k^2+4k+4\right) = \frac{(k+1)^2(k+2)^2}{4},$$

Therefore $1^3+\dots+k^3+(k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$, which is what we wanted to prove. $\boxtimes$

c) Prove $9 + 9\times10 + 9\times10^2 + \dots + 9\times10^{n-1} = 10^n - 1$.

Proof: By induction
Base Case: $n = 1$ $\quad 9 = 10^1 - 1 = 9$ ✓

Let's assume $9 + 9\times10 + \dots + 9\times10^{k-1} = 10^k - 1$
and let's prove $9 + 9\times10 + \dots + 9\times10^{k-1} + 9\times10^k = 10^{k+1} - 1$

Since $9 + 9\times10 + 9\times10^2 + \dots + 9\cdot10^{k-1} = 10^k - 1$
then $9 + 9\times10 + 9\times10^2 + \dots + 9\cdot10^{k-1} + 9\cdot10^k = 10^k - 1 + 9\cdot10^k$

$$= 10^k(1+9) - 1$$
$$= 10^k(10) - 1$$
$$= 10^{k+1} - 1 \quad \text{QED.}$$

d) Prove
$$\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \dots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

Proof: By induction, the base case is $n=1$

$$\frac{1}{1\cdot2} = \frac{1}{2}$$

$$1 - \frac{1}{1+1} = 1 - \frac{1}{2} = \frac{1}{2}$$

so the base case is correct.

Assume $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \dots + \frac{1}{k(k+1)} = 1 - \frac{1}{k+1}$

then
$$\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= 1 - \frac{(k+2)-1}{(k+1)(k+2)}$$

$$= 1 - \frac{k+1}{(k+1)(k+2)}$$

$$= 1 - \frac{1}{k+2}$$

which is what we wanted to prove.

e) Prove $1 + x + x^2 + \dots + x^n = \frac{1-x^{n+1}}{1-x}$

Proof: Base case: $n=0$

$$1 = \frac{1-x^1}{1-x} = 1 \checkmark$$

Assume $1+x+x^2+\dots+x^k = \frac{1-x^{k+1}}{1-x}$

let's check $n=k+1$

$$1+x+x^2+\dots+x^k+x^{k+1} = \frac{1-x^{k+1}}{1-x} + x^{k+1}$$

$$= \frac{1-x^{k+1}+(1-x)x^{k+1}}{1-x} = \frac{1-x^{k+1}+x^{k+1}-x^{k+2}}{1-x}$$

$$= \frac{1-x^{k+2}}{1-x}$$

so the $n=k$ case implies the $n=k+1$ case, so by induction $1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}$. ☒

f) Prove $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$

Proof: Base case: $n=0$

$$\lim_{x\to\infty} \frac{x^0}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0 \checkmark$$

Let's do another case to illustrate the process.

$n=1$.

$$\lim_{x\to\infty} \frac{x^1}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0 \quad \text{(by L'Hôspital).}$$

Assume it's true for $n=k$, that is

$$\lim_{x\to\infty} \frac{x^k}{e^x} = 0.$$

Let's check if it works for $n=k+1$.

$$\lim_{x\to\infty} \frac{x^{k+1}}{e^x} = \lim_{x\to\infty} (k+1)\frac{x^k}{e^x} \quad \text{by L'Hôspital}$$

Since $k+1$ is constant and $\lim_{x\to\infty} \frac{x^k}{e^x} = 0$

then $\lim_{x\to\infty} (k+1)\frac{x^k}{e^x} = 0$

Therefore $\lim_{x\to\infty} \frac{x^{k+1}}{e^x} = 0$

Therefore by induction $\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$ ▢

g) Prove $n! = \int_0^\infty x^n e^{-x}\, dx$

Proof: The base case is $n=0$.
Let's check. $0! = 1$

$$\int_0^\infty x^0 e^{-x}\, dx = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = -\cancel{e^{-\infty}}^{\,0} + e^{-0} = 1.$$

Let's assume it's true for $n=k$, i.e.

$$k! = \int_0^\infty x^k e^{-x}\, dx$$

Now $\int_0^\infty x^{k+1} e^{-x}\, dx = \ ?$

Let's do integration by parts.

$u = x^{k+1}$ $dv = e^{-x}\,dx$

$du = (k+1)\,x^k\,dx$ $v = -e^{-x}$

$$\int_0^\infty x^{k+1} e^{-x}\,dx = -x^{k+1}e^{-x}\Big|_0^\infty + \int_0^\infty (k+1)\,x^k e^{-x}\,dx$$

Since $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$

then $-x^{k+1}e^{-x}\Big|_0^\infty = 0$

so $\int_0^\infty x^{k+1} e^{-x}\,dx = \int_0^\infty (k+1)\,x^k e^{-x}\,dx$

$= (k+1)\int_0^\infty x^k e^{-x}\,dx$

Since $\int_0^\infty x^k e^{-x}\,dx = k!$

then $\int_0^\infty x^{k+1} e^{-x}\,dx = (k+1)\,k! = (k+1)!$

Therefore by induction $n! = \int_0^\infty x^n e^{-x}\,dx.$ $\square$

h) Prove that the nth derivative of x^n is $n!$

Proof: The zeroth derivative of $x^0 = 1$ is $1 = 0!$ ✓
The first derivative of x^1 is 1 and $1 = 1!$ ✓
So the base case is covered.
Suppose the k-th derivative of x^k is $k!$, i.e. $\frac{d^k}{dx^k} x^k = k!$
Let's now take the $(k+1)^{th}$ derivative of x^{k+1}.

$(x^{k+1})' = (k+1)\,x^k$ so

$$\frac{d^{k+1}}{dx^{k+1}}x^{k+1} = (k+1)\frac{d^k}{dx^k}x^k = (k+1)\,k! = (k+1)!$$

so by induction $\frac{d^n}{dx^n}x^n = n!$ $\boxtimes$

22.5 a) Prove $2^n \le 2^{n+1} - 2^{n-1} - 1$

Pf: Base case $n=1$

$$2^1 = 2$$
$$2^{1+1} - 2^{1-1} - 1 = 2^2 - 2^0 - 1 = 2$$

so $2 \le 2$ is true.

Induction Hypothesis: Suppose $2^k \le 2^{k+1} - 2^{k-1} - 1$

Goal: Prove $2^{k+1} \le 2^{k+2} - 2^k - 1$.

$$2^k \le 2^{k+1} - 2^{k-1} - 1$$

so
$$2^{k+1} = 2 \cdot 2^k \le 2\left(2^{k+1} - 2^{k-1} - 1\right)$$
$$= 2^{k+2} - 2^k - 2$$
$$< 2^{k+2} - 2^k - 1$$

Therefore $2^{k+1} \le 2^{k+2} - 2^k - 1$ $\boxtimes$

b) Prove $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{2^n}\right) \ge \frac{1}{4} + \frac{1}{2^{n+1}}$

Pf: Base case: $n = 1$ $\quad 1 - \frac{1}{2} = \frac{1}{2}$

$$\frac{1}{4} + \frac{1}{2^{1+1}} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{2} \ge \frac{1}{2}$$

so it's true for $n = 1$.

Induction Hypothesis: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^k}\right) \geq \frac{1}{4}+\frac{1}{2^{k+1}}$

Goal: Prove $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^{k+1}}\right) \geq \frac{1}{4}+\frac{1}{2^{k+2}}$

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^{k+1}}\right)$$

$$\geq \left(\frac{1}{4}+\frac{1}{2^{k+1}}\right)\left(1-\frac{1}{2^{k+1}}\right)$$

$$= \frac{1}{4}+\frac{1}{2^{k+1}}-\frac{1}{4\cdot 2^{k+1}}-\frac{1}{2^{k+1}\cdot 2^{k+1}}$$

$$= \frac{1}{4}+\frac{3}{4\cdot 2^{k+1}}-\frac{1}{2^{k+1}\cdot 2^{k+1}}$$

Since $k+1 \geq 2$, $2^{k+1} \geq 4$ so $-\frac{1}{2^{k+1}} \geq -\frac{1}{4}$

so

$$\frac{1}{4}+\frac{3}{4\cdot 2^{k+1}}-\frac{1}{2^{k+1}\cdot 2^{k+1}} \geq \frac{1}{4}+\frac{3}{4\cdot 2^{k+1}}-\frac{1}{4\cdot 2^{k+1}}$$

$$= \frac{1}{4}+\frac{2}{4\cdot 2^{k+1}}$$

$$= \frac{1}{4}+\frac{1}{2\cdot 2^{k+1}}$$

$$= \frac{1}{4}+\frac{1}{2^{k+2}}$$

so

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^{k+1}}\right) \geq \frac{1}{4}+\frac{1}{2^{k+2}} \quad \boxtimes$$

c) Prove

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$$

Pf: Base case $n = 1$

$$1 + \frac{1}{2} \geq 1 + \frac{1}{2}$$

Induction Hypotheses: $1 + \frac{1}{2} + \frac{1}{3} \dots + \frac{1}{2^k} \geq 1 + \frac{k}{2}$

Goal: Prove $1 + \frac{1}{2} + \frac{1}{3} \dots + \frac{1}{2^k} + \frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}} \geq 1 + \frac{k+1}{2}$

$$\left(1 + \frac{1}{2} + \frac{1}{3} \dots + \frac{1}{2^k}\right) + \frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}} \geq 1 + \frac{k}{2} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^{k+1}}$$

$$\frac{1}{2^k+1} \geq \frac{1}{2^{k+1}}$$

$$\frac{1}{2^k+2} \geq \frac{1}{2^{k+1}}$$

$$\vdots$$

$$\frac{1}{2^{k+1}} \geq \frac{1}{2^{k+1}}$$

so $1 + \frac{k}{2} + \frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}} \geq 1 + \frac{k}{2} + \underbrace{\frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \dots + \frac{1}{2^{k+1}}}_{\text{there are } 2^k \text{ of these}}$

since $2^{k+1} = 2^k + 2^k$

so
$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k} + \frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}} \geq 1 + \frac{k}{2} + \frac{2^k}{2^{k+1}}$$
$$= 1 + \frac{k}{2} + \frac{1}{2}$$
$$= 1 + \frac{k+1}{2}.$$

Therefore $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^{k+1}} \geq 1 + \frac{k+1}{2}$ ☒

d) Prove $\binom{2n}{n} < 4^n$.

Proof: By induction.

Base Case: For $n=1$ $\binom{2}{1} = 2$ $4^1 = 4$, $2 < 4$ ✓

Induction Hypothesis: Suppose $\binom{2k}{k} < 4^k$

Goal: Prove $\binom{2k+2}{k+1} < 4^{k+1}$.

$$\binom{2k+2}{k+1} = \frac{(2k+2)!}{(k+1)!(k+1)!} = \frac{(2k+2)(2k+1)(2k)!}{(k+1)k!(k+1)k!}$$

$$= \frac{(2k+2)(2k+1)}{(k+1)(k+1)}\binom{2k}{k} < \frac{(2k+2)(2k+1)}{(k+1)(k+1)}4^k.$$

Now $$\frac{(2k+2)(2k+1)}{(k+1)(k+1)} = \frac{2\cancel{(k+1)}(2k+2-1)}{\cancel{(k+1)}(k+1)} = \frac{2(2(k+1)-1)}{k+1}$$

$$= \frac{4(k+1)-2}{k+1} = 4 - \frac{2}{k+1} < 4$$

so $\frac{(2k+2)(2k+1)}{(k+1)(k+1)} < 4$. (Note: There are other ways to prove this)

so

$$\binom{2k+2}{k+1} < \frac{(2k+2)(2k+1)}{(k+1)(k+1)}4^k < 4 \cdot 4^k = 4^{k+1}$$

$\boxed{X}$

e) Prove $n! \leq n^n$

Proof: Base case: $n = 1$ $\quad 1! = 1$

$1^1 = 1$

so $1 \leq 1$.

Suppose $k! \leq k^k$

Let's prove $(k+1)! \leq (k+1)^{k+1}$

$$(k+1)! = (k+1)\,k! \leq (k+1)\,k^k$$

$$k < k+1$$

$$\text{so } k^k < (k+1)^k$$

$$\text{so } (k+1)\,k^k < (k+1)(k+1)^k = (k+1)^{k+1}$$

$$\text{so } (k+1)! < (k+1)^{k+1} \quad \boxtimes$$

f) Prove $1+2+3+\ldots+n \leq n^2$.

Proof: Base case $n=1$

$$1 \leq 1^2 \quad \checkmark$$

Induction Hypothesis: Suppose $1+2+\ldots+k \leq k^2$.

Goal: Prove $1+2+\ldots+k+(k+1) \leq (k+1)^2$.

$$1+2+\ldots+k+(k+1) \leq k^2+k+1 < k^2+2k+1 = (k+1)^2 \quad \boxtimes$$

22.7 Prove $\frac{1}{1^2}+\dots+\frac{1}{n^2} > \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)}$,

$$\frac{1}{1^2}+\dots+\frac{1}{n^2} \leq 1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{(n-1)n}$$

and conclude $1 \leq \zeta(2) \leq 2$

where $\zeta(2) = \lim_{n\to\infty} \frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{n^2}$.

Pf: Let's prove $\frac{1}{1^2}+\dots+\frac{1}{n^2} > \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)}$

Base case: $n=1$ $\quad \frac{1}{1^2} = 1 > \frac{1}{1\cdot 2}$.

Induction Hypothesis: Suppose $\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{k^2} > \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{k(k+1)}$

Goal: Prove $\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{k^2}+\frac{1}{(k+1)^2} > \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$

$$\left(\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{k^2}\right)+\frac{1}{(k+1)^2} > \left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{k(k+1)}\right)+\frac{1}{(k+1)^2}$$

since $(k+2)(k+1) > (k+1)^2$

$\Rightarrow \frac{1}{(k+1)^2} > \frac{1}{(k+2)(k+1)}$

so

$$\frac{1}{1^2}+\dots+\frac{1}{(k+1)^2} > \frac{1}{1\cdot 2}+\dots+\frac{1}{k(k+1)}+\frac{1}{(k+1)^2} > \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}.$$

Hence it is proven.

Now let's prove $\frac{1}{1^2}+\dots+\frac{1}{n^2} \leq 1+\frac{1}{1\cdot 2}+\dots+\frac{1}{(n-1)n}$

Base case: $n=1$: $\frac{1}{1^2} \leq 1$ ✓

I.H: Suppose $\frac{1}{1^2}+\dots+\frac{1}{k^2} \leq 1+\frac{1}{1\cdot 2}+\dots+\frac{1}{(k-1)k}$

Goal: Prove $\frac{1}{1^2}+\dots+\frac{1}{k^2}+\frac{1}{(k+1)^2} \leq 1+\frac{1}{1\cdot 2}+\dots+\frac{1}{(k-1)k}+\frac{1}{k(k+1)}$

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2} \leq 1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k-1)k}+\frac{1}{(k+1)^2}$$

$$< 1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k-1)k}+\frac{1}{k(k+1)}$$

Therefore by induction $\frac{1}{1^2}+\cdots+\frac{1}{n^2} \leq 1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$.

To finish we use that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)} = 1-\frac{1}{n+1}$

so

$$1-\frac{1}{n+1} < \frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n+1}$$

$$\zeta(2) = \lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)$$

so

$$1 \leq \zeta(2) \leq 2. \qquad \left(\text{since } \frac{1}{n+1}\to 0 \text{ as } n\to\infty\right)$$

$\boxtimes$