Homework 3

Solutions

(14.1) a) $R = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$

b) $R = \{(1,1), (2,1), (3,1), (4,1), (5,1), (2,2), (4,2), (3,3), (4,4), (5,5)\}$

c) $R = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$

d) $R = \{(1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5)\}$

(14.3) a) Reflexive, Symmetric, Anti-symmetric, Transitive
b) Anti-symmetric, Irreflexive
c) Anti-symmetric, Transitive
d) Symmetric,
e) Reflexive, Symmetric, Transitive

(14.5) Pf: Suppose xRy and yRx then $x=y$ and $y=x$ so in particular $x=y$.
This is true anytime xRy and yRx ~~so~~ R is anti-symmetric.

(14.6) a) $R = \{(x,y) \mid |x-y| \leq 2,\ x,y \in \mathbb{Z}\}$.

b) $(x,x) \in R$ because $|x-x| = 0 \leq 2$.
Therefore R is reflexive.

c) $(0,0) \in R$ hence R is NOT irreflexive.

d) Let $(x,y) \in R$. Then $|x-y| \leq 2$

Since $|y-x| = |x-y|$, then $|y-x| \leq 2$.
Therefore $(y,x) \in R$. Therefore R is symmetric.

e) R is not anti-symmetric because $1R2$ and $2R1$.

f) R is not transitive because $0R2$ and $2R4$ but $0 \not R 4$.

14.10 R is reflexive because if A is a finite subset of $\mathbb{Z}$, then $|A|=|A|$ so ARA.

R is symmetric because $ARB \Rightarrow |A|=|B|$ but $|A|=|B| \Rightarrow |B|=|A|$ so BRA.

R is not anti-symmetric because $\{1\} R \{2\}$ and $\{2\} R \{1\}$ yet $\{1\} \neq \{2\}$.

R is not irreflexive because $\{1\} R \{1\}$.

R is transitive because if ARB and BRC then ARC. Indeed if ARB and BRC then $|A|=|B|$ and $|B|=|C|$ so $|A|=|C|$ so ARC. $\boxtimes$

15.1 a) $N \mid 23-13=10$

so $N \mid 10$ and $N>1$ so $N=2, N=5$ or $N=10$.

b) $N \mid 10-5=5$ so $N=5$.

c) $N \mid 60-6=54$ so $N=2,3,6,9,18,27,54$

$54=2\cdot 3^3$

d) $N \mid 23-22=1$ so no $N>1$ makes the congruence true.

15.3 a) YES

b) NO (not reflexive)

c) NO (not symmetric)

d) No (not symmetric)

e) Yes

f) NO (not reflexive since it's missing $(4,4)$)

g) YES

15.7

a) $[1] = \{1, 2\}$

b) $[4] = \{4\}$

c) $[123] = \{120, 121, 122, 123, 124, 125, 126, 127, 128, 129\}$

d) $[me] = \{me, \text{my sister}\}$

e) $[me] = \{me, a_1, a_2 \ldots\}$ where $a_1, a_2 \ldots$ are people with the same birthday as me.

f) $[\{1,3\}] = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}\}$.

15.8

a) 10

b) 366

c) 8

d) 50

Partitions

16.1

$\{1,2,3\} \rightarrow$

$\{\{1\}, \{2\}, \{3\}\}$

$\{\{1,2\}, \{3\}\}$

$\{\{1,3\}, \{2\}\}$

$\{\{2,3\}, \{1\}\}$

$\{\{1,2,3\}\}$

$\{1,2,3,4\} \rightarrow \{\{1\}, \{2\}, \{3\}, \{4\}\}, \{\{1,2\}, \{3\}, \{4\}\}, \{\{1,3\}, \{2\}, \{4\}\},$
$\{\{1,4\}, \{2\}, \{3\}\}, \{\{2,3\}, \{1\}, \{4\}\}, \{\{2,4\}, \{1\}, \{3\}\},$
$\{\{3,4\}, \{1\}, \{2\}\}, \{\{1,2,3\}, \{4\}\}, \{\{1,2,4\}, \{3\}\},$
$\{\{1,3,4\}, \{2\}\}, \{\{2,3,4\}, \{1\}\}, \{\{1,2,3,4\}\},$
$\{\{1,2\}, \{3,4\}\}, \{\{1,3\}, \{2,4\}\}, \{\{1,4\}, \{2,3\}\}$.

16.10 There are 25! ways of placing $\{1,2,3,\ldots,25\}$ into the 25 squares and 5! of permuting the columns.

So there are $\boxed{\frac{25!}{5!}}$ different arrays.

(The relevent theorem is 16.6. You have an equivalence relation with equivalence classes of the same size. There are 25! elements of A and each equivalence class has 5! elements.)

16.15 For 4 elements is 7 (look at 16.1).

In general for $\{1,2,\ldots,n\}$ it's $2^{n-1}-1$.

So for $\{1,2,\ldots,100\}$ it's $\boxed{2^{99}-1}$

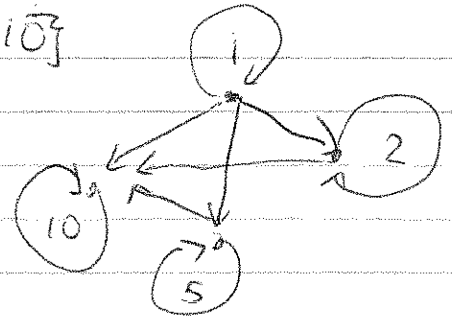
You can prove it by induction. Indeed it works for $n=1,2,3,4$. Suppose it works for $n=k$.

Consider The partitions of $\{1,2,3,\ldots,k+1\}$.

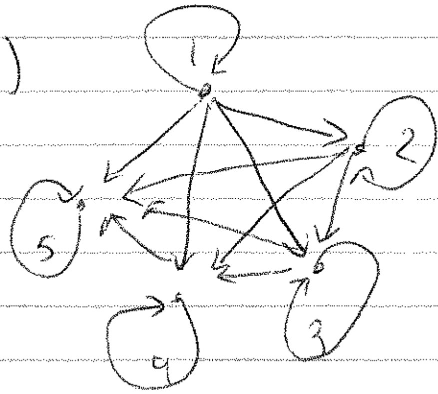
If you have a partition of $\{1,2,3,\ldots,k\}$ into 2 parts then you can put $k+1$ in one side or the other.
Also you can have the partition $\{\{1,2,3,\ldots,k\},\{k+1\}\}$
So # partitions it's $2(2^{k-1}-1)+1 = 2^k-1$.
The proof is complete by induction ☒

Another way is to say: 1 is on one side of the partition. 2 has 2 choices, 3 has 2 choices, …, n has 2 choices. So 2^{n-1} in total. But if everyone chose the same side it wouldn't be a partition in 2 parts, so you need to subtract 1.

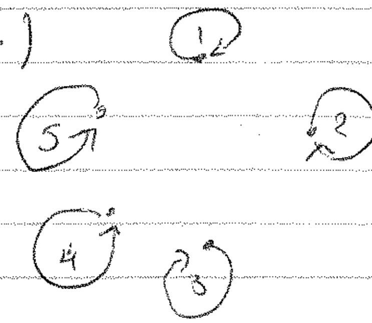
14.17 a) $A = \{1, 2, 5, 10\}$



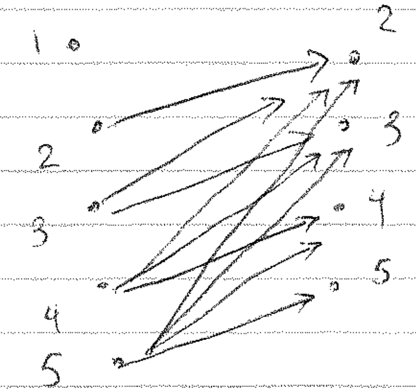
b)



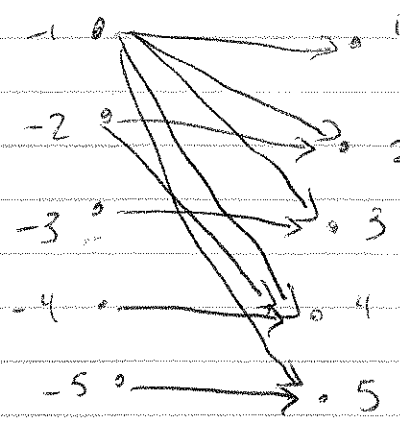
c)



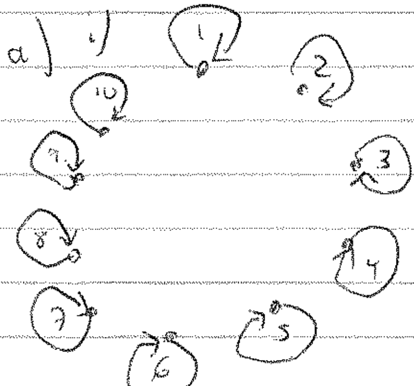
d)



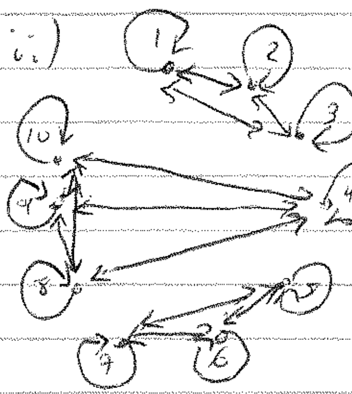
e)



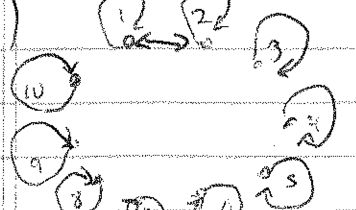
15.14 a) i)



ii)



iii)



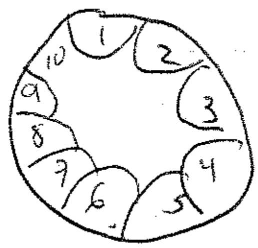
b) i) $[1]=\{1\}$, $[2]=\{2\}$, ..., $[10]=\{10\}$

ii) $[1]=[2]=[3]=\{1,2,3\}$, $[4]=[8]=[9]=[10]=\{4,8,9,10\}$, $[5]=[6]=[7]=\{5,6,7\}$

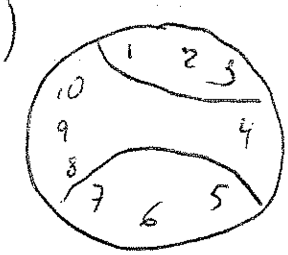
iii) $[1]=[2]=\{1,2\}$, $[3]=\{3\}$, $[4]=\{4\}$, ..., $[10]=\{10\}$.

c) Too ambiguous.

15.15 i)



ii)



iii)