Homework 6
SOLUTIONS

22.16 a) Prove $a_n = 2^{n+1} - 1$ where $a_0 = 1$ and $a_n = 2a_{n-1} + 1$ for $n \geq 1$.

Proof: Base case: $n = 0$.

$$a_0 = 1, \quad 2^{0+1} - 1 = 2^1 - 1 = \underline{1}$$

$$\text{so } a_0 = 2^1 - 1 \checkmark$$

I.H. Suppose $a_k = 2^{k+1} - \underline{1}$.
Goal: Prove $a_{k+1} = 2^{k+2} - 1$.

$$a_{k+1} = 2a_k + 1 = 2(2^{k+1} - 1) + 1 = 2^{k+2} - 2 + 1 = 2^{k+2} - \underline{1} \quad \boxed{\times}$$

b) Prove $b_n = \frac{3^n + 1}{2}$.

$b_0 = 1$ and $\frac{3^0 + 1}{2} = \underline{1}$ so the $n = 0$ case is true.

Suppose $b_k = \frac{3^k + 1}{2}$.

$$b_{k+1} = 3b_k - 1 = 3\left(\frac{3^k + 1}{2}\right) - 1 = \frac{3^{k+1} + 3 - 2}{2} = \frac{3^{k+1} + 1}{2} \quad \boxed{\times}$$

c) Prove $c_n = \frac{n^2 + n + 6}{2}$

$c_0 = 3$ and $\frac{0^2 + 0 + 6}{2} = 3$ so the $n = 0$ case is true

Suppose $c_k = \frac{k^2 + k + 6}{2}$.

$$c_{k+1} = c_k + k + 1 = \frac{k^2 + k + 6}{2} + k + 1 = \frac{k^2 + 3k + 8}{2} = \frac{k^2 + 2k + 1 + (k + 7)}{2}$$

$$= \frac{(k+1)^2 + (k+1) + 6}{2} \quad \boxed{\times}$$

d) Prove $d_n = 2^n + 3^n$

$n = 0$: $d_0 = 2^0 + 3^0 = 2 \checkmark$
$n = 1$: $d_1 = 2^1 + 3^1 = 5 \checkmark$

Suppose $d_k = 2^k + 3^k$ and $d_{k-1} = 2^{k-1} + 3^{k-1}$

$d_{k+1} = 5d_k - 6d_{k-1}$

$= 5(2^k+3^k) - 6(2^{k-1}+3^{k-1}) = 5\cdot 2^k + 5\cdot 3^k - 6\cdot 2^{k-1} - 6\cdot 3^{k-1}$

$= 2^{k-1}(10-6) + 3^{k-1}(15-6)$

$= 2^{k-1}(4) + 3^{k-1}(9) = 2^{k+1} + 3^{k+1}$ $\boxed{X}$

e) Prove $e_n = (n+1)2^n$.

Base cases $e_0 = (0+1)2^0 = 1$ ✓

$e_1 = (1+1)2^1 = 4$ ✓

Suppose $e_k = (k+1)2^k$. and $e_{k-1} = k\cdot 2^{k-1}$

$e_{k+1} = 4(e_k - e_{k-1}) = 4\left((k+1)2^k - k\cdot 2^{k-1}\right)$

$= 4\left(2^{k-1}\right)(2k+2-k)$

$= 2^{k+1}(k+2)$

$= ((k+1)+1)2^{k+1}$ $\boxed{X}$

f) Prove $F_n = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}$

Pf:

$F_0 = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}} = \dfrac{\sqrt{5}}{\sqrt{5}} = 1$ ✓

$F_1 = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} = \dfrac{\frac{6+2\sqrt{5}}{4} - \frac{6-2\sqrt{5}}{4}}{\sqrt{5}} = \dfrac{\frac{3+\sqrt{5}}{2} - \frac{3-\sqrt{5}}{2}}{\sqrt{5}} = \dfrac{2\sqrt{5}}{2\sqrt{5}} = 1$ ✓

Suppose $F_k = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$ and suppose $F_{k-1} = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}$

$F_{k+1} = F_k + F_{k-1} = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}} + \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}}{\sqrt{5}}$

$= \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{1-\sqrt{5}}{2}+1\right)}{\sqrt{5}} = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{3-\sqrt{5}}{2}\right)}{\sqrt{5}}$

$= \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}}{\sqrt{5}}$ $\boxed{X}$

22.17 A flagpole is n-th feet tall, There are 1 ft tall red flags, 2 ft tall blue flags and 2 ft tall green flags. Let g_n be the number of ways you can put the flags in the flagpole.

Prove $g_n = \frac{2}{3}\cdot 2^n + \frac{1}{3}(-1)^n$.

Proof: For $n=1$ $g_1 = 1$ (1 red flag)

and $\frac{2}{3}(2)^1 + \left(\frac{1}{3}\right)(-1)^1 = \frac{4}{3} - \frac{1}{3} = 1$ so $g_1 = 1$ ✓

For $n=2$, $g_2 = 3$ (3 red, 1 red + 1 blue, 1 red + 1 green)

also $\frac{2}{3}\cdot 2^2 + \left(\frac{1}{3}\right)(-1)^2 = \frac{8}{3} + \frac{1}{3} = 3$ ✓

Suppose $g_k = \frac{2}{3}2^k + \frac{1}{3}(-1)^k$ and $g_{k-1} = \frac{2}{3}2^{k-1} + \frac{1}{3}(-1)^{k-1}$.

Goal: Prove $g_{k+1} = \frac{2}{3}2^{k+1} + \frac{1}{3}(-1)^{k+1}$.

Consider the color of the first flag.

Three cases

First Flag	Feet Left	# Ways of placing flags
Red	k	g_k
Blue	k-1	g_{k-1}
Green	k-1	g_{k-1}

Therefore $g_{k+1} = g_k + 2g_{k-1}$

$$= \frac{2}{3}2^{k} + \frac{1}{3}(-1)^k + 2\left(\frac{2}{3}2^{k-1} + \frac{1}{3}(-1)^{k-1}\right)$$

$$= \frac{2}{3}2^k + \frac{1}{3}(-1)^k + \frac{2}{3}2^k + \frac{2}{3}(-1)^{k-1}$$

$$= \frac{2}{3}2^{k+1} + \frac{1}{3}(-1)^{k-1}(-1+2) = \frac{2}{3}2^{k+1} + \frac{1}{3}(-1)^{k-1}$$

(since $(-1)^{k+1} = (-1)^2(-1)^{k-1} = (-1)^{k-1}$)

$$= \frac{2}{3}2^{k+1} + \frac{1}{3}(-1)^{k+1}$$

$\boxtimes$

(20.4) a) For the sake of contradiction assume $A \subseteq B$, $B \subseteq C$ but $A \nsubseteq C$.

b) For the sake of contradiction suppose x and y are negative integers such that $x+y$ is NOT negative.

c) For the sake of contradiction suppose $x \in \mathbb{Q}$ and $x^2 \in \mathbb{Z}$ but $x \notin \mathbb{Z}$.

d) For the sake ~~of~~ of contradiction suppose p and q are primes such that $p+q$ is prime but neither p nor q is equal to 2.

e) For the sake of contradiction suppose that $\exists$ a line intersects all three sides of a triangle.

f) For the sake of contradiction suppose there exist 2 distinct circles that intersect at more than 2 points.

g) For the sake of contradiction, suppose there are finitely many primes.

(20.5) Prove that consecutive integers can't be both even.

Proof: Suppose $n \in \mathbb{Z}$ s.t n and $n+1$ are even.

Then $\exists\, a, b \in \mathbb{Z}$ s.t $n = 2a$ and $n+1 = 2b$.

Then $2b - 2a = 1$ so $b - a = \frac{1}{2}$. But $b-a$ is an integer, and $\frac{1}{2}$ is not. $\Rightarrow\Leftarrow$ $\boxtimes$

(Alternative proof: $n = 2a$ so $n+1 = 2a+1$ so $n+1$ is odd. Since a number cannot be both odd and even, we've reached a contradiction.)

(20.9) Prove $ab=0 \Rightarrow (a=0 \text{ or } b=0)$.

Proof: Suppose $ab=0$ but $a\neq 0$ and $b\neq 0$.

Since $a\neq 0$ and $b\neq 0$ then $ab\neq 0$. $\Rightarrow\Leftarrow$ $\boxtimes$

(20.10) Prove that for $a>1$, $1<\sqrt{a}<a$.

Proof: Suppose $a>1$ and $(\sqrt{a}\leq 1$ or $\sqrt{a}\geq a)$

If $\sqrt{a}\leq 1$ then since $\sqrt{a}>0$

~~we have~~ $(\sqrt{a})^2\leq 1^2$

so $a\leq 1$ $\Rightarrow\Leftarrow$

If $\sqrt{a}\geq a$ then since $\sqrt{a}>0$, $\frac{\sqrt{a}}{\sqrt{a}}\geq\frac{a}{\sqrt{a}}$ so $1\geq\sqrt{a}$

so by the previous case $a\leq 1$. $\Rightarrow\Leftarrow$

$\boxtimes$

(20.13) Prove $(A-B)\cap(B-A)=\emptyset$.

Pf: Suppose $(A-B)\cap(B-A)\neq\emptyset$. Then $\exists\, x\in(A-B)\cap(B-A)$

so $x\in A-B$ AND $x\in B-A$.

so $(x\in A$ and $x\notin B)$ AND $(x\in B$ and $x\notin A)$.

so $x\in A$ AND $x\notin A$ $\Rightarrow\Leftarrow$

Therefore $(A-B)\cap(B-A)=\emptyset$ $\boxtimes$

(21.3) Prove $n<2^n$ for all $n\in\mathbb{N}$.

Pf: Note $0<2^0=1$ and $1<2^1=2$.

Suppose A is the set of counterexamples. Since $A\subseteq\mathbb{N}$, there exists a least element in A, let's call it K.

Since 0 and 1 are not counterexamples $K\geq 2$.

Now k is the least counterexample and $k-1 \geq 1$ so $k-1 \in \mathbb{N}$, so $2^{k-1} > k-1$ (because $k-1$ is not a counterexample).

Since $2^{k-1} > k-1$ then $2^k > 2k-2$.

Since $k \geq 2$, $2k-2 = k+(k-2) \geq k+0 = k$

so $2^k > 2k-2 \geq k$.

But k is a counter example so $2^k < k$ $\Rightarrow \angle =$ $\boxtimes$

(21.7) Prove $F_n > (1.6)^n$ for n large enough (and find this n).

Proof: For $n = 29$, $F_{29} > (1.6)^{29}$ and for $n = 28$, $F_{28} < (1.6)^{28}$.

$F_{29} > (1.6)^{29}$ and $F_{30} > (1.6)^{30}$ are our base cases.

Suppose $F_k > (1.6)^k$ and $F_{k-1} > (1.6)^{k-1}$

then $F_{k+1} = F_k + F_{k-1} > (1.6)^k + (1.6)^{k-1} = (1.6)^{k-1}(1.6+1)$

$= (2.6)(1.6)^{k-1} > (2.56)(1.6)^{k-1}$

$= (1.6)^2(1.6)^{k-1} = (1.6)^{k+1}$

so $F_{k+1} > (1.6)^{k+1}$

so by induction $F_n > (1.6)^n$ whenever $n \geq$

(21.9) The mistake in the proof is that it doesn't verify if $x-3 \geq 0$, if $x-3 < 0$, $x-3$ is not in $\mathbb{N}$ and hence it could be a counterexample. $\boxtimes$