Solutions for the Poset worksheet

54.1

a) a and b are incomparable
b) $a < c$
c) c and g are incomparable
d) $b < h$
e) $c < i$
f) $h > d$

54.2

a) The height is 4.
There are several chains of length 4, here are a few of them:

$\{a, c, F, i\}$
$\{b, d, f, i\}$
$\{b, d, f, h\}$

(In the question you only needed to write one chain)

b) The width is 4.
Here are a couple of antichains of length 4:

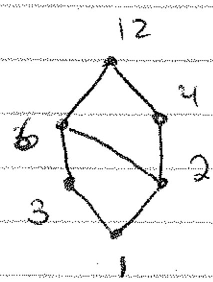
$\{e, c, d, g\}$
$\{e, c, b, g\}$.

c) $\{a, e, h\}$

d) $\{g, j\}$

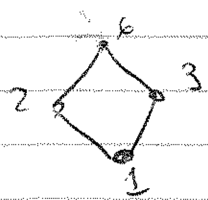
e) $\{h, i, j\}$

c

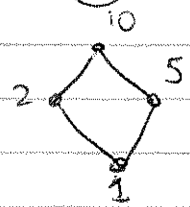


54.3

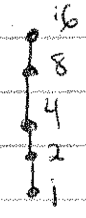
a



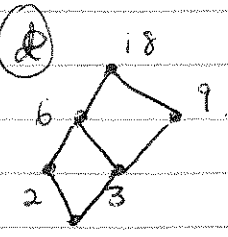
b



d



e



Note how the posets for $n=6$ and $n=10$ are isomorphic.

$\boxed{54.4}$ a) $\{1,2,6\}$, height $=3$
$\{2,3\}$, width $=2$

b) $\{1,2,10\}$, height $=3$
$\{2,5\}$, width $=2$

c) $\{1,3,6,12\}$, height $=4$
$\{2,3\}$, width $=2$

d) $\{1,2,4,8,16\}$, height $=5$
$\{1\}$, width $=1$

e) $\{1,2,6,18\}$, height $=4$
$\{2,3\}$, width $=2$

$\boxed{54.8}$ For the sake of contradiction suppose $x<y$ and $y<x$, then since $x<y$ we have $x \le y$
and since $y<x$ we have $y \le x$
so by antisymmetry $x=y$. But since $x<y$, $x \ne y$.
We've reached a contradiction so we can't have both $x<y$ and $y<x$.

(Caner's solution)
Alternative Solution: Suppose for the sake of contradiction that $x<y$ and $y<x$. So the poset restricted to $\{x,y\}$ is a linear order of size 2. So there exists a bijection $f: \{x,y\} \to \{1,2\}$ that is order-preserving. But then $f^{-1}(1) < f^{-1}(2)$ and $f^{-1}(2)$ is not $< f^{-1}(1)$. So one of $x<y$ and $y<x$ must be false.

$\boxed{55.1}$ The maximal elements are h, i, j.
The minimal elements are a, b, g.

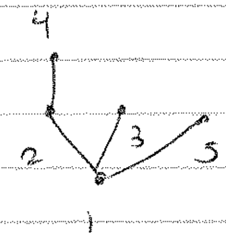
There are no maximums or minimums.

$\boxed{55.2}$

a)

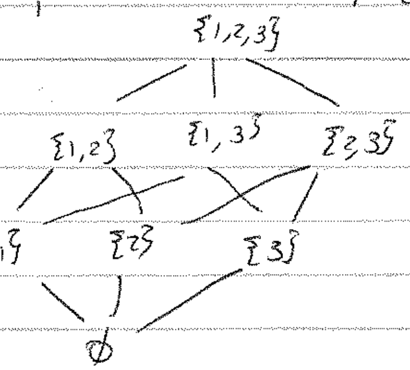
The Maximum and only maximal element is $\boxed{5}$.
The minimum and only minimal element is $\boxed{1}$.

b)



The maximals are $\boxed{4, 3, 5.}$
No maximum.
The minimum and only minimal is $\boxed{1}$.

c)



The maximum is $\boxed{\{1,2,3\}}$

The minimum is $\boxed{\emptyset}$

d) There are no maximals. Indeed suppose n is maximal. Since $n \mid 2n$ then $n \leq 2n$
but n is maximal so $2n \leq n$, so $n = 2n$ ⚡

Therefore n is not maximal.

[All prime numbers are minimal] because their only divisors are 1 and themselves and 1 is not in the poset.
Also, composites are not minimal, because any of their prime factors is below them.

e) The minimals are all people without children (alive).
The maximals are all people without living ancestors.

$\boxed{55.5}$

a) The maximum is $\{1,2,3,\ldots,n\}$.
The minimum is $\emptyset$.

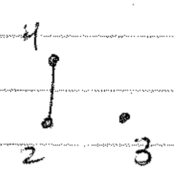
b) The maximals of this new subset are all $(n-1)$-element subsets of $\{1,2,3,\ldots,n\}$.
The minimals are all 1-element subsets of $\{1,2,3,\ldots,n\}$, i.e. $\{1\}, \{2\}, \{3\}, \ldots, \{n\}$.

$\boxed{55.7}$ a) Suppose M is a maximum. Suppose N is a maximum too.

Since M is a maximum, $\forall x \in X$, $M \geq X$.
In particular $M \geq N$.
But since N is a maximum $N \geq M$.
By antisymmetry $M = N$. Therefore there is only one maximum.

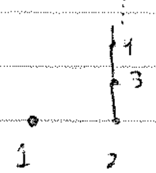
b) Yes. The poset $\{1\}$ with any relation is a poset with 1 maximum and minimum.

c) Yes. Consider the poset



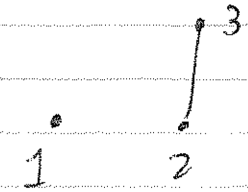
3 is maximal and minimal in that poset.

d) No. Example



1 is ~~the~~ the only maximal and not a maximum.

e) No.



1 is minimal
and 3 is maximal
but 1 and 3 are incomparable

f) Yes.

For a number to be a minimum is must be comparable with every other element of the poset. Since x it's not comparable to y, neither x nor y can be minimums.

g) Yes.

If they were comparable, i.e. if $x \leq y$ with x and y maximals, then by maximality $x \geq y$, so by antisymmetry $x = y$
~~but $x \neq y$~~.

$\boxed{56.1}$ 1

$\boxed{56.5}$ P and Q are isomorphic posets and f is the isomorphism.
Let $x \in P$.

a) x is minimum $\in P$ $\Leftrightarrow$ $f(x)$ is minimum in Q

pf: ($\Rightarrow$) Let x be the minimum in P. Since f is order-preserving

$$x \leq y \Rightarrow f(x) \leq f(y).$$

But $\forall y \in P$, $x \leq y$ so $\forall y \in P$, $f(x) \leq f(y)$

so $f(x) \leq f(y)$ $\forall f(y) \in Q$.

Since f is a bijection for any $q \in Q$
$\exists\ y \in P$ s.t $f(y) = q$. since $f(x) \leq f(y)$
$\Rightarrow f(x) \leq q$. Therefore $f(x)$
is the minimum.

$\Leftarrow$ The proof is the same but using the inverse of f.

b) The proof is analogous.

c) $(\Rightarrow)$ Let x be minimal in P. Let $y \in P$ be comparable to x. Then $x \leq y$ so $f(x) \leq f(y)$.

Let $z \in Q$. There exists $y \in P$ s.t $f(y) = z$.

Suppose z and $f(x)$ are comparable.

If $z \leq f(x) \Rightarrow f(y) \leq f(x) \Rightarrow y \leq x$

but x is minimal, so $y = x$.

If $z \geq f(x) \Rightarrow f(y) \geq f(x) \Rightarrow y \geq x$.

Indeed if x is minimal in $P \Rightarrow f(x)$ is minimal in Q.

$(\Leftarrow)$ Let $y \in P$. Let's prove that if y is comparable to x then $x \leq y$. If $y \geq x$ we're done, so suppose $y \leq x$.

Therefore $f(y) \leq f(x)$. But since $f(x)$ is minimal in Q, then $f(y) = f(x)$ so $x = y$ (since f is $\underline{1\text{-}1}$).

Therefore $x \leq y$, so x is minimal in P $\boxtimes$