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MATH 230 MIDTERM #1

September 30, 2013

INSTRUCTIONS: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- You may NOT use a calculator.
- Show all of your work.

Question	Points	Score
1	20	
2	20	
3	10	
4	20	
5	10	
6	10	
7	10	
8	10	
9	20	
10	10	
Total:	140	

Official Cheat Sheet

1. Let A be a set. Then 2^A is the set of all subsets of A. For example, if $A = \{1, 2\}$, then $2^A = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

2. $|A|$ is the number of elements of A. A useful formula is: $|A \cup B| = |A| + |B| - |A \cap B|$ if A and B are finite sets. Another useful formula is $\left|2^A\right| = 2^{|A|}$ when A is finite.

3. Here are some Boolean algebra properties (which can be translated easily to set properties by equating $\vee$ with $\cup$ and $\wedge$ with $\cap$):

   - $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.
   - $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$.
   - $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

4. $\mathbb{Z}$ is the set of integers. $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of positive integers.

5. Let A and B be sets. Then

   - $A \cup B = \{x | x \in A \text{ or } x \in B\}$,
   - $A \cap B = \{x | x \in A \text{ and } x \in B\}$,
   - $A - B = \{x | x \in A \text{ and } x \notin B\}$,
   - $A \Delta B = (A - B) \cup (B - A)$.
   - $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

6. Let a and b be integers.

   - a is even if there exists an integer c such that $a = 2c$.
   - a is odd if there exists an integer c such that $a = 2c + 1$.
   - We say $a|b$ (a divides b) if there exists an integer c such that $b = ac$.
   - a is composite if $|a| > 1$ and there exists c such that $1 < c < |a|$ and $c|a$.
   - a is prime if $a > 1$ and a is not composite.
   - a is perfect if a equals the sum of its positive divisors less than a.

1. True or False (Just answer true or false, you don't need to explain your answer):

   (a) [2 points] -23 is prime.

   FALSE (to be prime it would have to be positive)

   (b) [2 points] $7|1001$.

   TRUE ($\frac{1001}{7} = 143$)

   (c) [2 points] The sum of two odd numbers is odd.

   FALSE (The sum is even)

   (d) [2 points] $T \subseteq A$ if and only if $T \in 2^A$.

   TRUE (by definition)

   (e) [2 points] $\emptyset \subseteq \{\emptyset\}$.

   TRUE (the empty set is the subset of anything)

   (f) [2 points] Let $n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime. n is a perfect number.

   TRUE (Proved in class)

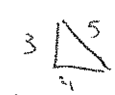
   (g) [2 points] $2 \in \{1, 2, \{1, 2\}\}$.

   TRUE

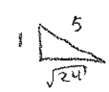
   (h) [2 points] If $x^2 < 0$, then x is a perfect number.

   TRUE (No x satisfies $x^2 < 0$, so it is a vacuous statement)

   (i) [2 points] Two right triangles that have hypotenuses of the same length have the same area.

   FALSE (Example 

has area 6

has area $\frac{\sqrt{24}}{2} = \sqrt{6}$)

   (j) [2 points] $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy = 1$.

   FALSE (For $x = 2$, there is no $y \in \mathbb{Z}$ because $\frac{1}{2}$ is not an integer)

2. For the following pairs of statements A, B, write a if the statement "If A, then B" is true, write b if the statement "If B, then A" is true and write c if the statement "A if and only if B is true". You should write all that apply.

(a) [5 points] A: $x > 0$. B: $x^2 > 0$.

a

(b) [5 points] A: Ellen is a grandmother. B: Ellen is female.

a

(c) [5 points] A: x is odd. B: $x + 1$ is even.

a, b, c

(d) [5 points] A: Polygon $PQRS$ is a rectangle. B: Polygon $PQRS$ is a square.

b

3. Proofs:

(a) [5 points] Using the definition of *odd* integer provided in the "cheat sheet", prove that if n is an odd integer, then $-n$ is also an odd integer.

Proof: Let n be an odd integer. Therefore there exists an integer c such that $n = 2c+1$.

$$\begin{aligned} -n &= -2c-1 \\ &= -2c-2+1 \\ &= 2(-c-1)+1 \\ &= 2d+1 \end{aligned}$$

where $d = -c-1$.

Therefore there exists an integer d such that $-n = 2d+1$.

Therefore $-n$ is an odd integer. $\boxtimes$

(b) [5 points] Let a, b and d be integers. Suppose $b = aq + r$ where q and r are integers. Prove that if $d|a$ and $d|b$, then $d|r$.

Proof: Let a, b, d, q, r be integers such that $b = aq + r$ and $d | a$ and $d | b$.

Since $d|a$ there exists an integer m such that $a = dm$

Since $d|b$ there exists an integer n such that $b = dn$

Therefore

$$\begin{aligned} r &= b - aq \\ &= dn - (dm)q \\ &= d(n - mq) \\ &= dc \end{aligned}$$

for $c = n - mq$.

Therefore there exists an integer c such that $r = dc$.

Therefore $d|r$. $\boxtimes$

4. Find counterexamples to disprove the following statements:

(a) [5 points] An integer x is positive if and only if $x+1$ is positive.

$x=0$ is a counterexample since $x+1=1$ is positive, yet 0 is not positive.

(b) [5 points] An integer is a *palindrome* if it reads the same forwards and backwards when expressed in base 10. For example, 1331 is a *palindrome*. All *palindromes* are divisible by 11.

101 is a palindrome, yet it is not divisible by 11, so it is a counterexample.

(c) [5 points] If a, b and c are positive integers then $a^{(b^c)} = (a^b)^c$.

$a=2,\ b=3,\ c=2$

$$2^{(3^2)} = 2^9 \qquad (2^3)^2 = 2^6 \qquad 2^9 \neq 2^6,$$

so $a=2, b=3, c=2$ is a counterexample.

(d) [5 points] Let A, B and C be sets. Then $A-(B-C) = (A-B)-C$.

Let $A=\{1,2\}$, $B=\{3\}$, $C=\{2,3\}$.

$$A-(B-C) = \{1,2\} - (\{3\}-\{2,3\}) = \{1,2\} - \emptyset = \{1,2\}$$

$$(A-B)-C = (\{1,2\}-\{3\}) - \{2,3\} = \{1,2\} - \{2,3\} = \{1\}.$$

As easily seen, this is a counter example since $A-(B-C) \neq (A-B)-C$ in this example.

5. Boolean Algebra

(a) [5 points] Prove or disprove that the Boolean expressions $x \to \neg y$ and $\neg(x \to y)$ are logically equivalent.

x	y	$\neg y$	$x \to \neg y$	$x \to y$	$\neg(x \to y)$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	T	T	F
F	F	T	T	T	F

They are not logically equivalent since $x=y=F$ yields $x \to \neg y = T$ while $\neg(x \to y) = F$

(b) [5 points] The expression $x \to y$ can be rewritten in terms of only the basic operations $\wedge, \vee$ and $\neg$; that is, $x \to y = (\neg x) \vee y$. Find an expression that is logically equivalent to $x \leftrightarrow y$ that uses only the operations $\wedge, \vee, \neg$ and prove that your expression is correct.

$$x \leftrightarrow y = \boxed{((\neg x) \vee y) \wedge (x \vee (\neg y))}$$

x	y	$x \leftrightarrow y$	$((\neg x) \vee y) \wedge (x \vee (\neg y))$
T	T	T	$(F \vee T) \wedge (T \vee F) = T \wedge T = T$
T	F	F	$(F \vee F) \wedge (T \vee (T)) = F \wedge T = F$
F	T	F	$(T \vee T) \wedge (F \vee F) = T \wedge F = F$
F	F	T	$(T \vee F) \wedge (F \vee T) = T \wedge T = T$

Indeed, they are logically equivalent.

Another possibility that works is $\boxed{(x \wedge y) \vee ((\neg x) \wedge (\neg y))}$

6. Consider the following proposition. Let N be a two-digit number and let M be the number formed from N by reversing the digits of N. Now compare N^2 and M^2. The digits of M^2 are precisely those of N^2, but reversed. For example:

$$10^2 = 100 \qquad 01^2 = 001$$

$$11^2 = 121 \qquad 11^2 = 121$$

$$12^2 = 144 \qquad 21^2 = 441$$

$$13^2 = 169 \qquad 31^2 = 961$$

and so on. Here is a proof of the proposition:

Proof. Since N is a two-digit number, we can write $N = 10a + b$ where a and b are the digits of N. Since M is formed from N by reversing digits, $M = 10b + a$.

Note that $N^2 = (10a + b)^2 = 100a^2 + 20ab + b^2 = (a^2) \times 100 + (2ab) \times 10 + (b^2) \times 1$, so the digits of N^2 are, in order, $a^2, 2ab, b^2$.

Likewise, $M^2 = (10b + a)^2 = (b^2) \times 100 + (2ab) \times 10 + (a^2) \times 1$, so the digits of M^2 are, in order, $b^2, 2ab, a^2$, exactly the reverse of N^2, which completes the proof.

(a) [5 points] Prove that the proposition is false.

$15^2 = 225$

$51^2 = 2500 + 100 + 1 = 2601$

225 and 2601 are not just reversing the digits.

(b) [5 points] Explain why the proof is invalid.

$a^2, 2ab, b^2$ are digits if they are numbers less than or equal to 9.

If a or b is at least 4 then a^2, b^2 are greater than 9, so they are not digits. The proof doesn't take this into account.

7. Counting

(a) [5 points] In how many ways can we make a list of three integers (a, b, c) where $0 \leq a, b, c \leq 9$ such that $a + b + c$ is even?

a has 10 choices (any number)
b has 10 choices (any number)
c has 5 choices (an even number if a+b is even, an odd number if a+b is odd).

$\boxed{10 \times 10 \times 5 = 500.}$

(b) [5 points] In how many ways can we make a list of three integers (a, b, c) where $0 \leq a, b, c \leq 9$ such that abc is even?

$\boxed{10^3 - 5^3}$

because for abc not to be even a, b, c must be odd. There are 10^3 lists (a, b, c) and 5^3 of them have a, b, c all odd.

8. Evaluate the following:

(a) [5 points] $\displaystyle\prod_{k=1}^{100} \frac{k+1}{k}$.

$$\frac{2 \cdot 3 \cdot \ldots \cdot 101}{1 \cdot 2 \cdot \ldots \cdot 100} = \boxed{101}$$

(b) [5 points] $\displaystyle\prod_{k=0}^{100} \frac{k^2}{k+1}$.

$$\frac{0^2 \cdot 1^2 \cdot 2^2 \cdot \ldots \cdot 100^2}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 100 \cdot 101} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 100}{101} = \boxed{\frac{100!}{101}}$$

9. Let $A \times B = \{(1,2),(1,3),(1,7),(2,2),(2,3),(2,7)\}$.

(a) [5 points] What is $A \cup B$?

$A = \{1, 2\}$
$B = \{2, 3, 7\}$

$\boxed{A \cup B = \{1, 2, 3, 7\}}$

(b) [5 points] What is $A \cap B$?

$\boxed{A \cap B = \{2\}}$

(c) [5 points] What is $A - B$?

$A - B = \{1\}$

(d) [5 points] What is $A \Delta B$?

$(A - B) \cup (B - A)$

$= \{1\} \cup (\{3, 7\}) = \boxed{\{1, 3, 7\}}$

10. [10 points] Let A, B and C be sets. Prove that

$$(A \cup B) - C = (A - C) \cup (B - C).$$

Proof: ~~Let~~ $(\Rightarrow)$

Let $x \in (A \cup B) - C$

then $x \in A \cup B$ and $x \notin C$

then $x \in A$ or $x \in B$ and $x \notin C$

Therefore $(x \in A$ and $x \notin C)$ or $(x \in B$ and $x \notin C)$

Therefore $x \in (A - C) \cup (B - C)$.

$(\Leftarrow)$ Let $x \in (A - C) \cup (B - C)$

Then $x \in (A - C)$ or $x \in (B - C)$

so $(x \in A$ and $x \notin C)$ or $(x \in B$ and $x \notin C)$

so $(x \in A$ or $x \in B)$ and $x \notin C$

so $x \in (A \cup B) - C$ $\boxtimes$

Alternative proof using Boolean algebra.

Let $m = x \in A$, $n = x \in B$, $r = x \notin C$.

In general

$$(m \wedge r) \vee (n \wedge r) = (m \vee n) \wedge r \quad \text{(Distributive Property)}$$

So $\big((x \in A) \wedge (x \notin C)\big) \vee \big((x \in B) \wedge (x \notin C)\big) = \big((x \in A) \vee (x \in B)\big) \wedge (x \notin C)\big)$

Therefore ~~A~~ $(A - C) \cup (B - C) = (A \cup B) - C$ $\boxtimes$