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MATH 230 MIDTERM #2

March 3, 2014

INSTRUCTIONS: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- You may NOT use a calculator.
- Show all of your work.

Question	Points	Score
1	20	
2	20	
3	10	
4	20	
5	15	
6	15	
7	10	
Total:	110	

1. True or False (Just answer true or false, you don't need to explain your answer):

   (a) [2 points] Let R be an equivalence relation on the set A. Let $a \in A$, then $a \in [a]$.

   TRUE

   (b) [2 points] Let R be an equivalence relation on A. If $a \in A$ and $b \in A$, then the intersection of $[a]$ and $[b]$ is empty.

   FALSE, if aRb then $[a]=[b]$.

   (c) [2 points] Let R be a symmetric relation. Then $R = R^{-1}$.

   TRUE

   (d) [2 points] If a relation is symmetric then it is not antisymmetric.

   FALSE (the "is-equal-to" relation is both)

   (e) [2 points] There is a relation that is reflexive and irreflexive.

   TRUE (the empty relation on the empty set).

   (f) [2 points] If a relation is reflexive and symmetric it must also be transitive.

   FALSE

   (g) [2 points] A relation R on a set A is antisymmetric if and only if $R \cap R^{-1} \subseteq \{(a,a) : a \in A\}$

   TRUE

   (h) [2 points] Let R be an equivalence relation on a set A. The equivalence classes of R form a partition of the set A.

   TRUE (by definition)

   (i) [2 points] The difference between a proof by strong induction and a proof by induction is that the base cases are dealt with differently.

   FALSE (the difference is in the hypothesis)

   (j) [2 points] The is-less-than-or-equal-to relation is an equivalence relation.

   FALSE (not symmetric)

2. Prove that the following are true:

(a) [5 points] $1+3+3^2+\ldots+3^n = \frac{3^{n+1}-1}{2}$, for all positive integers n.

For $n=1$: $1+3=4$ and $\frac{3^2-1}{2}=4$ so it's true for $n=1$.

Suppose $1+3+\cdots+3^k = \frac{3^{k+1}-1}{2}$, ~~t~~

Now $1+3+\cdots+3^k+3^{k+1} = \frac{3^{k+1}-1}{2}+3^{k+1} = \frac{3^{k+1}-1+2\cdot 3^{k+1}}{2} = \frac{3\cdot 3^{k+1}-1}{2}$

$= \frac{3^{k+2}-1}{2}$.

Since $1+3+\cdots+3^{k+1} = \frac{3^{k+2}-1}{2}$, then the statement is true by induction.

(b) [5 points] $1+5+9+\ldots+(4n-3) = 2n^2-n$, for all positive integers n.

For $n=1$:

$1 = 2(1^2)-1 = 1$ ✓

Suppose $1+5+9+\ldots+(4k-3) = 2k^2-k$.

Prove $1+5+9+\cdots+(4k-3)+(4k+1) = 2(k+1)^2-(k+1)$

$= 2(k^2+2k+1)-k-1$

$= 2k^2+3k+1$.

$(1+5+9+\ldots+(4k-3))+(4k+1) = 2k^2-k+4k+1$

$= 2k^2+3k+1$

So the LHS equals the RHS.
The proof is complete.

(c) [5 points] $n^2 > 2n + 1$, for all integers $n \geq 3$.

For $n=3$, the LHS is 9 and the RHS is 7

$9 > 7$ ✓

Suppose for some $k \geq 3$ $k^2 > 2k+1$.

Let's prove $(k+1)^2 > 2(k+1)+1 = 2k+3$

$(k+1)^2 = k^2+2k+1 > (2k+1)+2k+1 = 4k+2$

We want $4k+2 > 2k+3$ so we need $2k > 1$

but $2k > 1$ since $k \geq 3$.

Therefore $(k+1)^2 > 2k+3$ and the proof is complete.

(d) [5 points] $2^n > n^2$, for all integers $n \geq 5$.

Base Case: $2^5 = 32 > 25 = 5^2$.

Suppose $2^k > k^2$ for some $k \geq 5$.

$2^{k+1} = 2 \cdot 2^k > 2k^2 = k^2 + k^2$.

We want to prove $2^{k+1} > (k+1)^2 = k^2 + (2k+1)$

so it comes down to showing that

$k^2 + k^2 > k^2 + (2k+1)$ so to

showing that $k^2 > 2k+1$ for $k \geq 5$.

From (c) we know $k^2 > 2k+1$ for $k \geq 3$,

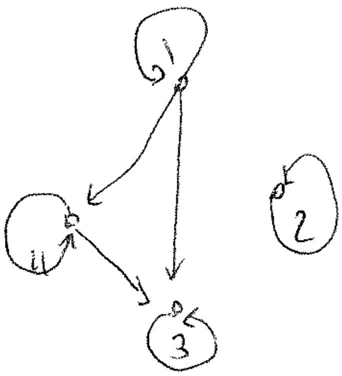
therefore $k^2 > 2k+1$ for $k \geq 5$ so

$2^{k+1} > (k+1)^2$ $\boxed{X}$

3. For each of the following relations defined on the set $\{1, 2, 3, 4\}$ determine whether they are reflexive, irreflexive, symmetric, antisymmetric and/or transitive.

(a) [5 points]

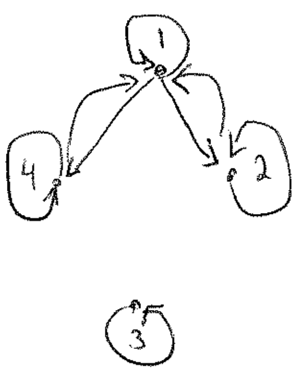
$$R = \{(1,1), (2,2), (3,3), (1,4), (4,4), (1,3), (4,3)\}.$$



Reflexive, Antisymmetric, Transitive

(b) [5 points]

$$R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (4,4), (1,4), (4,1)\}.$$



Reflexive, Symmetric

Not transitive because $(4,1) \in R \wedge (1,2) \in R$
but $(4,2) \notin R$.

4. For each equivalence relation below, find the requested equivalence classes.

(a) [5 points] $R = \{(x, y) : x, y \in \mathbb{Z} \text{ and } |x| = |y|\}$. Find $[-3]$ and $[0]$.

$[-3] = \{-3, 3\}$

$[0] = \{0\}$

(b) [5 points] R is has-the-same-size-as relation on $2^{\{1,2,3,4,5\}}$. Find $[\{1, 2, 3\}]$.

$\{ \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\} \}$

(c) [5 points] R is the relation

$$R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(4,4),(1,4),(4,1),(2,4),(4,2)\}$$

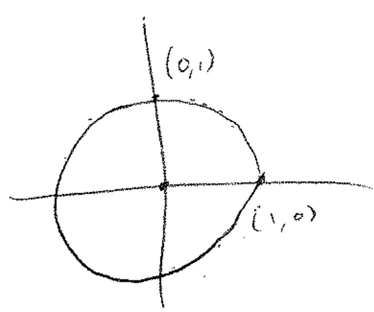
on the set $\{1,2,3,4\}$. Find [1] and [2].

$[1] = \{1, 2, 4\}$

$[2] = \{1, 2, 4\}$

(d) [5 points] R is the relation on the set of points on the plane where the point (a, b) is related to the point (c, d) if and only if $a^2+b^2 = c^2+d^2$ (for all a, b, c, d real numbers). What figure in the plane does $[(0,1)]$ represent?

The unit circle.



5. [15 points] Let R be the relation on the set of points on the plane where the point (a, b) is related to the point (c, d) if and only if $a^2+b^2=c^2+d^2$ (for all a, b, c, d real numbers). Prove that R is an equivalence relation on the set of points on the plane.

To prove it we must prove R is reflexive, symmetric and transitive.

Reflexive:

$(a,b) R (a,b)$ because $a^2+b^2=a^2+b^2$

so R is reflexive.

Symmetric: Suppose $(a,b) R (c,d)$.

Then $a^2+b^2=c^2+d^2$

So $c^2+d^2=a^2+b^2$

so $(c,d) R (a,b)$

so R is symmetric.

Transitive: Suppose $(a,b) R (c,d)$ and $(c,d) R (e,f)$

then $a^2+b^2=c^2+d^2$ and $c^2+d^2=e^2+f^2$

Therefore $a^2+b^2=e^2+f^2$

So $(a,b) R (e,f)$

so R is transitive.

So R is an equivalence relation.

6. Number of relations

(a) [5 points] Let $A = \{1\}$. How many different relations on A are there?

2 ($\emptyset$ and $\{(1,1)\}$).

(b) [5 points] Let $A = \{1, 2\}$. How many different relations on A are there?

$\boxed{2^4 = 16.}$

There are 4 possible ~~relations~~ pairs ((1,1), (1,2), (2,1), (2,2)) and each one could be in a relation or not.

(c) [5 points] Let $A = \{1, 2, 3 \ldots, n\}$. How many different relations on A are there?

2^{n^2}.

There are n^2 pairs from A. Each pair can be in the relation or not.

7. A special type of door lock has a panel with five buttons labeled with the digits 1 through 5. This lock is opened by a sequence of three actions. Each action consists of either pressing one of the buttons or pressing a pair of them simultaneously. For example, 12-4-3 is a possible combination. The combination 12-4-3 is the same as 21-4-3 because both the 12 and the 21 simply mean to press buttons 1 and 2 simultaneously.

   (a) [5 points] How many combinations are possible?

Each step has 15 possibilities (the 5 single digit possibilities plus the $\binom{5}{2} = 10$ 2-digit possibilities)

So $15 \times 15 \times 15 = \boxed{15^3}$

   (b) [5 points] How many combinations are possible if no digit is repeated in the combination?

There are 3 possibilities: 1) 2 2-digit combos and 1 1-digit
2) 1 2-digit combo and 2 1-digits
3) 3 1-digit combos

select 2 digits, select 2 digits from remaining

1) $3\binom{5}{2}\binom{3}{2}\binom{1}{1} = 90$ ways (multiply by 3 to select which step is the 1-digit combo)

2) $3\binom{5}{2}\binom{3}{1}\binom{2}{1} = 180$ ways (multiply by 3 to select the 2-digit combo)

3) $\binom{5}{1}\binom{4}{1}\binom{3}{1} = 60$ ways.

Total # of possibilities is $90 + 180 + 60 = \boxed{330}$