Name:____________________________

Math 230 Midterm #3

April 4, 2014

Instructions: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- You may NOT use a calculator.
- Show all of your work.

Question	Points	Score
1	20	
2	10	
3	10	
4	20	
5	10	
6	10	
7	10	
Total:	90	

1. True or False:
   For the following suppose that A, B and C are sets.
   (a) [2 points] Every function is a relation.

   (b) [2 points] Every relation is a function.

   (c) [2 points] The Pigeonhole principle can be stated as: "Let A and B be finite sets and let $f : A \to B$. If $|A| > |B|$, then f is not one-to-one."

   (d) [2 points] The function $f = \{(1,1),(2,1),(3,4)\}$ is a function $f : \{1,2,3,4\} \to \{1,4\}$.

   (e) [2 points] $f(x) = 7x - 12$ is a bijection from $(0,1)$ to $(-12,-5)$.

   (f) [2 points] If $f : A \to B$ is one-to-one and $g : B \to C$ is one-to-one, then $g \circ f$ is one-to-one.

   (g) [2 points] Let $f : A \to B$. If f is one-to-one, then f^{-1} is a function and $f^{-1} : B \to A$.

   (h) [2 points] If $f : A \to B$ is a bijection, then $f \circ f^{-1} = id_A$.

   (i) [2 points] If $f = id_A$, $g = id_B$ such that $A \subseteq B$ and $A \neq B$, then $f \circ g$ is undefined.

   (j) [2 points] If there are 13 people in a room, then at least two of them were born on the same month (not necessarily on the same year).

2. For each of the following statements, write the first sentences of a proof by contradiction:

   (a) [2 points] If a square of a rational number is an integer, then the rational number must also be an integer.

   (b) [2 points] Distinct circles intersect in at most two points.

   (c) [2 points] If the sum of two primes is prime, then one of the primes must be 2.

   (d) [2 points] There are infinitely many primes.

   (e) [2 points] Consecutive integers cannot both be even.

3. Prove the following statements:

   (a) [5 points] If the sum of two primes is prime, then one of the primes must be two.

   (b) [5 points] Let A and B be sets such that $A \cap B = \emptyset$. Then $(A \times B) \cap (B \times A) = \emptyset$.

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = |x|$ and let $g : \mathbb{N} \to \mathbb{N}$ be defined by $g(x) = |x|$.

   (a) [5 points] Prove or disprove: f is one-to-one.

   (b) [5 points] Prove or disprove: f is onto.

(c) [5 points] Prove or disprove: g is one-to-one.

(d) [5 points] Prove or disprove: g is onto.

5. (a) [5 points] Prove that if $n \geq 10^{10}$ is a positive integer, then two of its digits must be the same.

(b) [5 points] The squares of an 8×8 chess board are colored black or white (not necessarily the same way a usual chess board is colored). We call a group of squares an L-region if it consists of a corner square, the two squares above it and the two squares to its right (so it has the shape of an L with equal width and height). Prove that no matter how we color the chess board, there must be two L-regions that are colored identically.

6. [10 points] Let $A = \{1, 2, 3, 4, 5\}$ with $f : A \to A$, $g : A \to A$, and $h : A \to A$. We are given the following:

- $f = \{(1, 2), (2, 3), (3, 1), (4, 3), (5, 5)\}$,
- $h = \{(1, 3), (2, 3), (3, 2), (4, 5), (5, 3)\}$, and
- $h = f \circ g$.

Find all possible functions g that satisfy these conditions.

7. [10 points] Let A, B and C be sets. Prove that if $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f$ is a bijection.