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MATH 230 MIDTERM #3

April 4, 2014

INSTRUCTIONS: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- You may NOT use a calculator.
- Show all of your work.

Question	Points	Score
1	20	
2	10	
3	10	
4	20	
5	10	
6	10	
7	10	
Total:	90	

1. True or False:
For the following suppose that A, B and C are sets.

T (a) [2 points] Every function is a relation.

F (b) [2 points] Every relation is a function.

T (c) [2 points] The Pigeonhole principle can be stated as: "Let A and B be finite sets and let $f : A \to B$. If $|A| > |B|$, then f is not one-to-one."

F (d) [2 points] The function $f = \{(1,1),(2,1),(3,4)\}$ is a function $f : \{1,2,3,4\} \to \{1,4\}$.

$\mathrm{dom}\, f = \{1,2,3\} \neq \{1,2,3,4\}$

T (e) [2 points] $f(x) = 7x - 12$ is a bijection from $(0,1)$ to $(-12,-5)$.

T (f) [2 points] If $f : A \to B$ is one-to-one and $g : B \to C$ is one-to-one, then $g \circ f$ is one-to-one.

F (g) [2 points] Let $f : A \to B$. If f is one-to-one, then f^{-1} is a function and $f^{-1} : B \to A$.

$f^{-1} : \mathrm{Im} f \to A$ if f is not onto f^{-1} is not from B to A.

F (h) [2 points] If $f : A \to B$ is a bijection, then $f \circ f^{-1} = id_A$.

$f \circ f^{-1} = id_B$.

T (i) [2 points] If $f = id_A$, $g = id_B$ such that $A \subseteq B$ and $A \neq B$, then $f \circ g$ is undefined.

T (j) [2 points] If there are 13 people in a room, then at least two of them were born on the same month (not necessarily on the same year).

2. For each of the following statements, write the first sentences of a proof by contradiction:

   (a) [2 points] If a square of a rational number is an integer, then the rational number must also be an integer.

   For the sake of contradiction suppose $r \in \mathbb{Q}$ and $r^2 \in \mathbb{Z}$ yet $r \notin \mathbb{Z}$.

   (b) [2 points] Distinct circles intersect in at most two points.

   For the sake of contradiction suppose 2 circles intersect in more then 2 points.

   (c) [2 points] If the sum of two primes is prime, then one of the primes must be 2.

   For the sake of contradiction suppose p, q and $p+q$ are prime and $p \neq 2$ and $q \neq 2$.

   (d) [2 points] There are infinitely many primes.

   For the sake of contradiction suppose there are finitely many primes.

   (e) [2 points] Consecutive integers cannot both be even.

   For the sake of contradiction suppose two consecutive integers are even.

3. Prove the following statements:

(a) [5 points] If the sum of two primes is prime, then one of the primes must be two.

Suppose p, q and $p+q$ are prime.
For the sake of contradiction suppose $p \neq 2$ and $q \neq 2$.
Since $p \neq 2$ and $q \neq 2$ and p, q are prime then p and q are odd. Since they are odd, then $p+q$ is even. Since $p+q$ is even and $p+q$ is prime, $p+q=2$. But $p, q \geq 3$ so $p+q \geq 6$ $\Rightarrow\Leftarrow$

Therefore one of p or q must be 2.

(b) [5 points] Let A and B be sets such that $A \cap B = \emptyset$. Then $(A \times B) \cap (B \times A) = \emptyset$.

Let $A \cap B = \emptyset$.
Suppose $(A \times B) \cap (B \times A) \neq \emptyset$
Therefore $\exists$ $(a,b) \in (A \times B) \cap (B \times A)$.
Since $(a,b) \in A \times B$ then $a \in A$ and $b \in B$.
Since $(a,b) \in B \times A$ then $a \in B$ and $b \in A$.

Therefore $a \in A$ and $a \in B$ so $a \in A \cap B$. $\Rightarrow\Leftarrow$

Therefore $(A \times B) \cap (B \times A) = \emptyset$.

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = |x|$ and let $g : \mathbb{N} \to \mathbb{N}$ be defined by $g(x) = |x|$.

(a) [5 points] Prove or disprove: f is one-to-one.

Since $f(2) = f(-2)$, then f is not $\underline{1\text{-}1}$.

(b) [5 points] Prove or disprove: f is onto.

Since $f(x) \neq -1$ for any x, f is <u>not</u> onto

(c) [5 points] Prove or disprove: g is one-to-one.

Suppose $g(x) = g(y)$.

Then $|x| = |y|$.

Since $x \in \mathbb{N}$ then $|x| = x$

Since $y \in \mathbb{N}$ then $|y| = y$.

Therefore $x = y$.

Therefore g is one to one.

(d) [5 points] Prove or disprove: g is onto.

Let $y \in \mathbb{N}$.

Let $x = y$.

Then since $x \in \mathbb{N}$, $f(x) = |x| = \cancel{x} = y$.

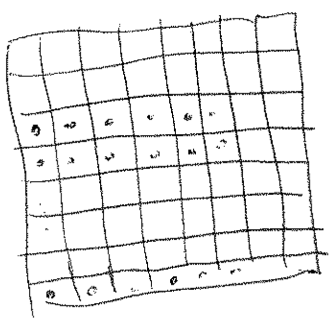
Therefore $f(x) = y$.

Since y is arbitrary, f is onto.

5. (a) [5 points] Prove that if $n \geq 10^{10}$ is a positive integer, then two of its digits must be the same.

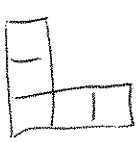
Since $n \geq 10^{10}$, n has at least 11 digits. Therefore By Pigeonhole Principle at least two of them must match.

(b) [5 points] The squares of an 8×8 chess board are colored black or white (not necessarily the same way a usual chess board is colored). We call a group of squares an L-region if it consists of a corner square, the two squares above it and the two squares to its right (so it has the shape of an L with equal width and height). Prove that no matter how we color the chess board, there must be two L-regions that are colored identically.



The corner of the L can be placed in 36 spots

Therefore there are 36 possible L-regions.



contains 5 squares so there are $2^5 = 32$ ways to color it.

Since there are 36 L's and 32 coloring configurations, by the Pigeonhole Principle at least 2 L's have the same coloring configuration.

6. [10 points] Let $A = \{1,2,3,4,5\}$ with $f : A \to A$, $g : A \to A$, and $h : A \to A$. We are given the following:

- $f = \{(1,2),(2,3),(3,1),(4,3),(5,5)\}$,
- $h = \{(1,3),(2,3),(3,2),(4,5),(5,3)\}$, and
- $h = f \circ g$.

Find all possible functions g that satisfy these conditions.

$h = f \circ g$. $h(1) = 3$ so $f(g(1)) = 3$

so $g(1) = 2$ or $g(1) = 4$.

$h(2) = 3$ so $g(2) = 2$ or $g(2) = 4$

$h(3) = 2$ so $f(g(3)) = 2$ so $g(3) = 1$.

$h(4) = 5$ so $f(g(4)) = 5$ so $g(4) = 5$.

$h(5) = 3$ so $g(5) = 2$ or $g(5) = 4$.

There are eight possible g's:

$g = \{(1,2), (2,2), (3,1), (4,5), (5,2)\}$

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7. [10 points] Let A, B and C be sets. Prove that if $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f$ is a bijection.

We have to prove $g \circ f$ is onto and one-to-one.

Let's start by proving that it is onto.

Note $g \circ f : A \to C$, so let $c \in C$.

Since g is onto there exists $b \in B$ such that $g(b) = c$.

Since f is onto and $b \in B$, there exists $a \in A$ s.t $f(a) = b$.

Therefore $g \circ f(a) = g(f(a)) = g(b) = c$, so

$$g \circ f(a) = c.$$

Therefore $g \circ f$ is onto.

Let's prove $g \circ f$ is 1-1.

Suppose $g \circ f(x) = g \circ f(y)$.

then $g(f(x)) = g(f(y))$.

Since g is 1-1, then

$$f(x) = f(y).$$

Since f is 1-1, then

$$x = y.$$

Therefore $g \circ f$ is one-to-one.

Since $g \circ f$ is onto and one-to-one, $g \circ f$ is a bijection.