P_2Q_3 and P_3Q_2; let S be the intersection of P_1Q_3 and P_3Q_1; and let T be the intersection of P_1Q_2 and P_2Q_1. Then R, S, and T are collinear. (See Figure 4.4.)

Pappus' theorem is really a special case of Pascal's theorem, since two lines may be thought of as a degenerate conic.

Exercise 4.34. Write a script which demonstrates Pascal's theorem for a circle.

Theorem 4.5.3 (Desargues' Theorem). *Let P be a point not on a triangle ΔABC. Let A', B', and C' be points on the lines PA, PB, and PC, respectively, as in Figure 4.5. Let the (extended) sides BC and $B'C'$ meet at R. Similarly, let AC and $A'C'$ meet at S and let AB and $A'B'$ meet at T. Then R, S, and T are collinear.*

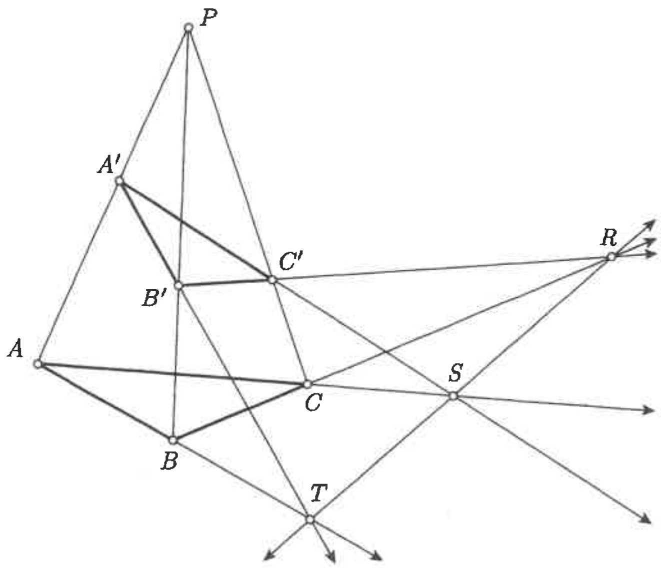

Figure 4.5

Exercise 4.35. Write a script which demonstrates Desargues' theorem.

The proof of Desargues' theorem is actually not too difficult, if thinking in three dimensions comes easily.

Proof of Desargues' Theorem. In Figure 4.5, think of the rays emanating from P as the edges of a pyramid. Then, ΔABC can be thought of as the intersection of a plane α with this pyramid. Similarly, $\Delta A'B'C'$ is the intersection of a plane α' with this pyramid.

The intersection of the two planes α and α' is a line l. Since the line BC is on α, and $B'C'$ is on α', the intersection of these two lines must be in the intersection of the two planes. That is, R is on l. Similarly, both S and T are on l. That is, the three points are collinear. □

Exercise 4.36. Let $ABCDEF$ be a hexagon inscribed in a circle. Let AB and CD intersect at G, and let DE and AF intersect at H. Write a script which shows that BE, CF, and GH are coincident.

Exercise 4.37. State and prove the converse of Desargues' theorem. [H]

4.6 Parabola Paper

In this section, we construct *parabola paper* (see Exercise 3.39). In Geometer's Sketchpad, select 'Plot Points' under the 'Graph' pull-down menu. Manually insert the points (x, x^2) in .1 increments from .1 to 1.6. That is, enter (.1, .01), (.2, .04), ..., (1.6, 2.56). Notice that the points (0, 0) and (1, 0) are already plotted. Select and move the point (1, 0) to stretch the axis system. Join the points with line segments and hide the points (but not the points (0, 0) or (1, 0)). Select the y-axis, mark it as a mirror by double clicking it (or select it and choose 'Mark Mirror' under the 'Transform' menu). Select all (under the 'Edit' menu), and reflect the graph using 'Reflect' under the 'Transform' menu. Hide the axis and any leftover stray marks, leaving only the parabola and the points (0, 0) and (1, 0). Save the Sketch for use in the following exercises.

Exercise 4.38*. Use the parabola paper to construct a regular 7-gon (see Exercise 3.39).

Exercise 4.39*. Use the parabola paper to construct a regular 9-gon.

Exercise 4.40.** Use the parabola paper to construct a regular 13-gon.

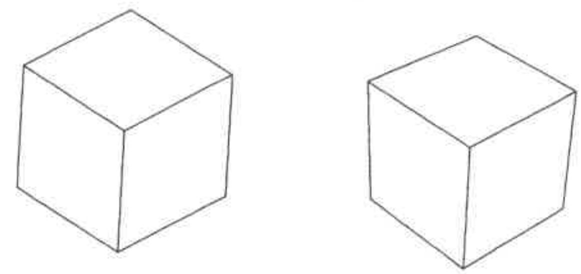

Figure 11.2. A cube, projected onto a plane both perpendicularly and through a point.

of the main ideas of projective geometry is the idea that we should not be constrained to lines not parallel to **A**. So let us extend **A** to include a *line at infinity*, which corresponds to all lines parallel to **A**. The plane **A** together with the line at infinity is called the *projective plane* and we denote it with $\mathbb{P}^2$.

A line in **A** corresponds to a plane through O, so we call the intersection of $\mathbb{P}^2$ with a plane through O a *line* in $\mathbb{P}^2$. Note that every pair of distinct planes through O intersect in a line through O. Thus, every pair of distinct lines in $\mathbb{P}^2$ intersect at exactly one point. We should therefore not be surprised that projective geometry and elliptic geometry are related. Note that the intersection of a sphere **S** centered at O and a line through O is a pair of antipodal points on **S**. In the elliptic geometry **P**, we identified antipodal points on a sphere **S**, so there exists a one-to-one correspondence between points in **P** and points in $\mathbb{P}^2$. A line in **P** corresponds to a great circle on **S**, which can be thought of as the intersection of **S** with a plane through O. Thus, there is also a one-to-one correspondence between lines in **P** and lines in $\mathbb{P}^2$. Since **P** includes a metric, there is a natural way to induce a metric on $\mathbb{P}^2$, but there is actually a lot to be gained by resisting this temptation.

Note that, on **P**, there is no distinguished line. Thus, when we think of $\mathbb{P}^2$ as being the plane **A** together with a line at infinity, we should not give this line any undue importance. It is just like any other line.

The plane **A** may be thought of as a Euclidean plane without a metric. When thought of this way, we call it the *affine plane*.

11.1 Moving a Line to Infinity

Sometimes, carefully chosen definitions can be very powerful because of the way they make us think about things. So far, we have only defined projective geometry, and it appears all we have is Euclidean geometry together with a line at infinity and without a metric. The proof of the following theorem illustrates just how powerful and how much more our definitions actually contain. This theorem is known as Pappus' theorem, and was introduced in Section 4.5 as an example of a result which we were not yet ready to prove but could be nicely demonstrated with Sketchpad. We are now ready to prove this result.

Theorem 11.1.1 (Pappus' Theorem). *Let P_1, P_2, and P_3 be three points on the line l_1, and let Q_1, Q_2, and Q_3 be three points on the line l_2. Let R be the intersection of P_2Q_3 and P_3Q_2; let S be the intersection of P_1Q_3 and P_3Q_1; and let T be the intersection of P_1Q_2 and P_2Q_1. Then R, S, and T are collinear (see Figure 11.3).*

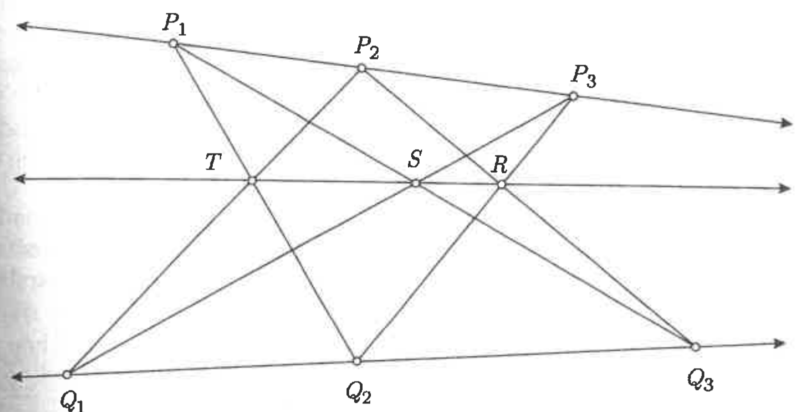

Figure 11.3

Proof. Let us first suppose we have such a diagram on a plane **A** embedded in three dimensions. Pick a point O not on **A**, and think of **A** as a subset of $\mathbf{P}^2$. Let us think of each point and line on **A** as being lines and planes through O respectively. Let l be the line through R and S. We want to show that T lies on l. Let the line l corresponds to a plane L through O, and let T correspond to the line t through O. We therefore want to show that t lies on L.

Let us now consider a different plane $\mathbf{A}'$ not passing through O. The intersection of $\mathbf{A}'$ with the lines and planes induced by the original diagram create a different diagram on $\mathbf{A}'$. Label these new points P_1', P_2', etc. Note that T lies on l if and only if T' lies on l' in the diagram on $\mathbf{A}'$. Thus, it is enough to prove the result for the diagram on $\mathbf{A}'$. Our idea is to orient $\mathbf{A}'$ in such a way that the diagram on $\mathbf{A}'$ is more convenient to work with.

Let us choose $\mathbf{A}'$ so that $\mathbf{A}'$ and L are parallel. Then the line l' is the line at infinity, so R' and S' are at infinity. That is, the lines $P_2'Q_3'$ and $P_3'Q_2'$ are parallel, as are the lines $P_1'Q_3'$ and $P_3'Q_1'$. Hence, we get the convenient diagram pictured in Figure 11.4. The point T' is at infinity if and only if $P'_1Q'_2$ and $P'_2Q'_1$ are parallel, so let us now prove that.

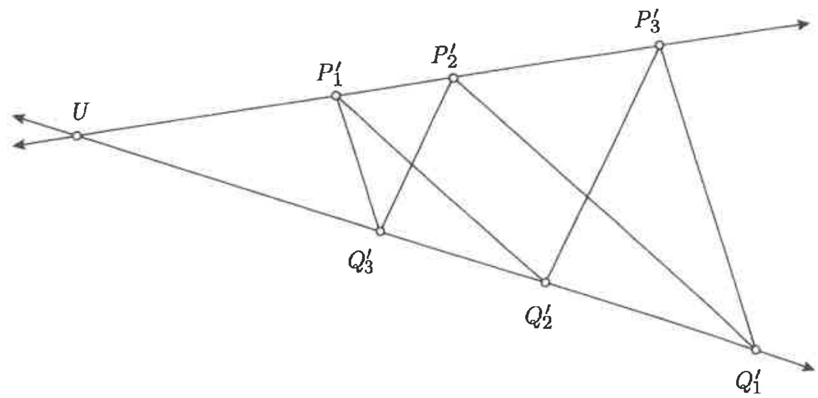

Figure 11.4

Let l'_1 and l'_2 intersect at U. Let us think of $\mathbf{A}'$ as a Euclidean plane, on which there exists a notion of distance. Since $P'_2Q'_3$ and $P'_3Q'_2$ are parallel, we know

$$\frac{|UP'_2|}{|UQ'_3|} = \frac{|UP'_3|}{|UQ'_2|},$$

and since $P'_1Q'_3$ and $P'_3Q'_1$ are parallel, we get

$$\frac{|UP'_1|}{|UQ'_3|} = \frac{|UP'_3|}{|UQ'_1|}.$$

Combining these two, we get

$$\frac{|UP'_1|}{|UQ'_2|} = \frac{|UP'_2|}{|UQ'_1|},$$

from which it follows that $P'_1Q'_2$ and $P'_2Q'_1$ are parallel. Thus, T' is on the line at infinity, so the line t is on the plane L, so T lies on the line l, as desired. □

The idea of choosing $\mathbf{A}'$ so that it is parallel to L is called *moving a line to infinity*. This technique can also be used to prove Desargues' theorem, which was introduced in Section 4.5 too.

Theorem 11.1.2 (Desargues' Theorem). *Let P be a point not on ΔABC. Let A', B', and C' be points on the lines PA, PB, and PC, respectively. Let the (extended) sides BC and $B'C'$ meet at R. Similarly, let AC and $A'C'$ meet at S and let AB and $A'B'$ meet at T. Then R, S, and T are collinear (see Figure 4.5).*